# Exercises for the lattice and EFT Summer School

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The hadron resonance gas model is a simple model which is often used to interpret data both from lattice QCD simulations and heavy ion collision experiments. In this exercise sheet you will use this model to estimate the QCD equation of state below the crossover temperature, the curvature of the crossover line in the baryochemical potential, and the severity of the sign problem at small $\mu_B$.

The model approximates the free energy az a sum over hadron resonances, treated as free particles:

$$\log \mathcal{Z}^{HRG}(T, \mu_B, \mu_S, \mu_Q) = \sum_i \log \mathcal{Z}_i(T, \mu_B, \mu_S, \mu_Q)$$
$$\log \mathcal{Z}_i^{HRG} = \eta g_i V \int \frac{d^3p}{(2\pi)^3} \log\left(1 + \eta \lambda_i e^{-\sqrt{m_i^2+p^2}/T}\right)$$
$$\lambda_i = e^{B_i\mu_B + Q_i\mu_Q + S_i\mu_S}$$

where $\eta_i = -1$ for baryons and 1 for mesons, $m_i$ is the mass, $B_i, S_i$ and $Q_i$ are the baryon number, strangeness and electric charge of the given resonance and $g_i$ is the spin degeneracy.

**a)** You can use the provided list of hadron resonances from the 2015 PDG. Baryons and antibarions are also listed, all isospin states are listed. The columns in the file are:

1. particle id
2. name
3. mass in GeV
4. Spin degeneracy
5. Baryon number
6. Strangeness
7. Electric charge

Write a program that can caculate the pressure and the energy density for a given value of $T$, $\mu_B$, $\mu_S$ and $\mu_Q$. You can either do the integrals numerically, or use the following (quickly converging) series expansion:

$$\log \mathcal{Z}_i = \frac{g_i}{2\pi^2} V T m_i^2 \sum_{k=1}^{\infty} (-\eta_i)^k \frac{\lambda_i^k}{k^2} K_2\left(\frac{k m_i}{T}\right),$$

where $K_2$ is a modified Bessel function of the second kind. Compare the model result at zero chemical potentials to the provided continuum QCD equation of state (the pressure $p = T \log \mathcal{Z}/V$ and energy density $\epsilon = Ts - p + \sum_{i=B,S,Q} \mu_i n_i$ as a function of $T$). What is the energy density where the HRG prediction first goes $1\sigma$ away from the lattice results? We will use this value as a proxy for crossover conditions.

**b)** Write a subroutine that can calculate the conditions for strangeness neutrality: For fixed $T$ and $\mu_B$ it should find the value of $\mu_S$ where the strangeness vanishes in the model.

**c)** Calculate the curvature of the line in the $T - \mu_B$ plane where the energy density is equal to the value you found in point a). Compare this with the lattice determinations of the curvature of the crossover line in the lectures.

**d)** Show that for QCD with 2 degenerate flavours the variance of the phase of the determinant ($\mu_u = \mu_d = \mu_B/3$) is given by:

$$\left\langle \left( \frac{\det M_u}{|\det M_u|} \right)^2 \right\rangle = -\frac{4}{9} \chi_{11}^{ud} (LT)^3 \left( \frac{\mu_B}{T} \right)^2 + \mathcal{O}(\mu_B^4),$$

Assuming a (wrapped) Gaussian distribution for the angle $\theta$, what is then the ration of the full to the phase quenched partition function?

**e)** Change from the basis of $\mu_B, \mu_S$ and $\mu_Q$ to $\mu_u, \mu_d$ and $\mu_s$ by using the known charges of the different quarks and calculate $\chi_{11}^{ud}$ as a function of temperature in the hadron resonance gas model as a function of temperature. Which hadron gives the dominant contribution? Based on this, would you expect the sign problem at a fixed physical volume and temperature to be milder or more severe in the continuum vs on a coarse staggered lattice? Finally, look at Fig.6. of 1507.04627 [hep-lat] to confirm your guess.

**f)** The Boltzmann approximation of the HRG is when the sum over the $K$ Bessel functions is truncated at $k = 2$. For everything other than the pion, it is a good approximation. Can you contruct a quantity where the $\Omega$ baryon contributes in the Boltzmann approximation, but mesons and also baryons with less than 3 strange quarks do not?