Polylogarithms from meromorphic generating functions: Schottky numerics and kernel identities

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Elliptics and Beyond '24, MIAPbP
August 6^{th}, 2024

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Particle physicists task:

$$\mathcal{I} = \frac{\Gamma\left(\nu - \frac{\ell D}{2}\right)}{\prod_{j=1}^{n}\Gamma\left(\nu_j\right)} \int_{x_j \geq 0} \delta\left(1 - \sum_{j=1}^{n} x_j\right) \left(\prod_{j=1}^{n} \mathrm{d}x_j x_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\nu - \frac{(\ell+1)D}{2}}}{\mathcal{F}^{\nu - \frac{\ell D}{2}}}$$

- Accommodate scales, parameters and symmetries of your favourite physics problem in a special function?
- What is the class of such functions, what is a basis within this class? To what extend can the representation be chosen to be canonical?
- First steps towards canonicalization:
  use iterated integrals
  use integration kernels carrying the symmetries of the algebraic curve in question.
- Convenient: associate some symmetries of the problem with symmetries of the kernels.

Polylogarithms

Definition:

$$\mathrm{PolyLog}(z) = \int_{z_0}^{z} \omega_1(z_1)\, dz_1 \int_{z_0}^{z_1} \omega_2(z_2)\, dz_2 \cdots \int_{z_0}^{z_{n-1}} \omega_n(z_n)\, dz_n$$

- $z_0, z_i \in \Sigma$, $\Sigma \ldots$ orientable compact Riemann surface
- ***integration kernels*** $\omega(z)$ implement the geometry of the manifold Σ (e.g. acquire a phase when the argument is shifted by a cycle of the manifold)
- set of differential forms $\omega(z)dz \leftrightarrow$ ***class*** of functions.
- need to fix ***basepoint*** z_0 and regularization prescription accordingly
- Nice to have: ***generating function*** for the kernels ω_i.

Need to find mathematical language accomodating all analytic and algebraic constraints.
Need to find a representation which allows for numerical evaluation.

Polylogarithms ...

- ... nicely represent branch cut behaviour of QFT observables.
- Most special functions: representable as polylogs.
- Special values of polylogarithms: multiple zeta values, Euler sums.

Technical:

- simple algebraization:
  Hopf algebra, coaction/symbol $\rightarrow$ functional relations.
- algebraic tools allow to select basis for cohomology of iterated integrals of given depth.
- simple poles in differentials $\Leftrightarrow$ ϵ-form of Fuchsian differential equations.

Outline

- Polylogarithms on Riemann surfaces::general
  Enriquez' connection::definition
- Schottky uniformization::definition and practicabilities
  Enriquez' connection in the Schottky language
  Numerical benefits
- Identities for Enriquez' generating function
  Kernel identities::completeness and uniqueness
  Example

Polylogarithms again ... mathematically

- construct polylogarithms on a particular Riemann surface (compact, locally complex manifold)
- need to integrate along path. Path can be either straight (not picking up any monodromies) or wrap around certain cycles. Result should have a controllable behaviour when adding another wrapping.
- need connection form, built from a generating function, which can be expanded into various "wrapping configurations".
- keep information in a homological way: expand in a succession of wrappings around certain cycles: letter alphabet, free algebra
- need to decide about poles: {simple-pole | multi-pole} and associated residues
- need to decide about {meromorphic,quasiperiodic | fully periodic} kernels

Here: meromorphic, quasiperiodic generating function, with simple poles.

- Σ: Riemann surface of genus h
- homology basis: $\mathfrak{A}_i, \mathfrak{B}_i, i \in \{1 \dots h\}$ with holomorphic differentials ω_i with

$$\oint_{\mathfrak{A}_i} \omega_j = \delta_{ij}, \quad \oint_{\mathfrak{B}_i} \omega_j = \tau_{ij}.$$

- $\mathfrak{g}$: free Lie algebra generated by $a_i, b_i, i \in \{1 \dots h\}$

Enriquez' connection: [Enriquez]

$$K(z,x) = \sum_{r=0}^{\infty} \sum_{i_1,\cdots,i_r,j=1}^{h} \omega_{i_1 \cdots i_r j}(z,x) b_{i_1} \cdots b_{i_r} a_j$$

Properties:

$$K(\gamma_i z, x) = e^{b_i} K(z,x), \quad \forall i \in \{1 \dots h\},$$

$$(2\pi i) \operatorname{Res}_{z=x} K(z,x) = -\sum_{j=1}^{h} b_j a_j,$$

γ_i: move around the i-th $\mathfrak{B}$-cycle, $x, z \in \Sigma$ (suitable uniformization)

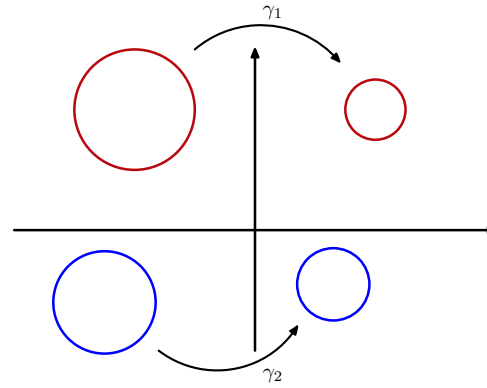
The above properties fix the function $K(z,x)$ uniquely. [Enriquez] [Zerbini]

Schottky uniformization and numerics for polylogarithms

with Konstantin Baune, Egor Im,
Artyom Lisitsyn and Federico Zerbini

Schottky parametrization

- for each genus: pair of circles C_i, C_i', Schottky generator $\gamma_i \in \mathrm{PSL}(2,\mathbb{C})$ such that $\gamma_i(C_i) = C_i'$
- fundamental domain for Riemann surface Σ: outside of all circles
- γ_i, $i \in 1,\ldots,h$ generate classical Schottky group G (Kleinian group, discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$)

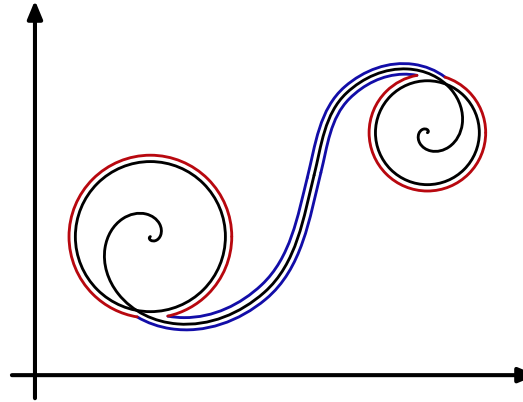


Define cosets:

$$G/G_i = \left\{\gamma_{j_1}^{n_1}\cdots\gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\right\},$$
$$G_i\backslash G = \left\{\gamma_{j_1}^{n_1}\cdots\gamma_{j_k}^{n_k} : \gamma_{j_1} \neq \gamma_i\right\}.$$

Loxodromies...

correspond to B-cycles:



Define fixed points for each generator:

$$P_i := \lim_{n\to\infty} \gamma_i^{-n} z \text{ and } P_i' := \lim_{n\to\infty} \gamma_i^{n} z \quad \text{for generic } z.$$

and express Abelian differentials:

$$\omega_i(z \mid G) = \frac{1}{2\pi i} \sum_{\gamma \in G/G_i} \left(\frac{1}{z - \gamma P_i'} - \frac{1}{z - \gamma P_i} \right) \mathrm{d}z\,.$$

Kronecker function::higher genus

Starting point: Kronecker function at genus one in Schottky parametrization: [Bernard]

$$S(z,x,w \mid G) = \frac{1}{2\pi i}\sum_{\gamma\in G}\frac{d(\gamma z)}{\gamma z - x}w^{-n} = \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}\frac{d(\gamma_1^n z)}{\gamma_1^n z - x}w^{-n}$$

Two Schottky uniformizations $\{C_i, C_i', \gamma_i\}$ and $\{\sigma(C_i), \sigma(C_i'), \sigma\gamma_i\sigma^{-1}\}$ for $\sigma \in \mathrm{PSL}(2,\mathbb{C})$ describe the same surface. Correspondingly, want:

$$S(z,x,w \mid G) = S\left(\sigma z, \sigma x, \sigma w \mid \sigma G \sigma^{-1}\right)$$

Extend to Mœbius-invariant expression:

$$S(z,x,w \mid G) := \sum_{\gamma\in G}\frac{\mathrm{d}(\gamma z)}{\gamma z - x}\frac{P_1 - x}{P_1 - \gamma z}W\left(\gamma^{-1}\right) = \sum_{\gamma\in G}\frac{\mathrm{d}z}{z-\gamma x}\frac{\gamma P_1 - \gamma x}{\gamma P_1 - z}W(\gamma).$$

with

$$W : \gamma_1^n \mapsto w^n\,.$$

... extend to higher genus:

$$S_j(z,x,\vec{w}\mid G) := \sum_{\gamma\in G} \frac{\mathrm{d}(\gamma z)}{\gamma z - x}\frac{P_j - x}{P_j - \gamma z} W\left(\gamma^{-1}\right) = \sum_{\gamma\in G} \frac{\mathrm{d}z}{z-\gamma x}\frac{\gamma P_j - \gamma x}{\gamma P_j - z} W(\gamma)$$

where $W : \gamma_{i_1}^{n_1}\cdots\gamma_{i_s}^{n_s} \mapsto w_{i_1}^{n_1}\cdots w_{i_s}^{n_s}$.

There is now a Kronecker form for each $j \in \{1...h\}$.

Properties:

Residue:

$$\mathrm{Res}_{z=x}\, S_j\left(\gamma_k z, x, \vec{w}\mid G\right) = 1.$$

Quasiperiodicities:

$$S_j\left(\gamma_k z, x, \vec{w}\mid G\right) = w_k S_j(z,x,\vec{w}\mid G)$$

Periodicity for pole location:

$$S_j\left(z,\gamma_k x,\vec{w}\mid G\right) = S_k(z,x,\vec{w}\mid G)\left(w_k^{-1}-1\right) + S_j(z,x,\vec{w}\mid G).$$

Equal to the connection by Enriquez by uniqueness ($w_k = e^{b_k}$) [Enriquez] [Zerbini]

Averaging and degeneration:

Each component of the Kronecker form is an average over genus-zero differentials:

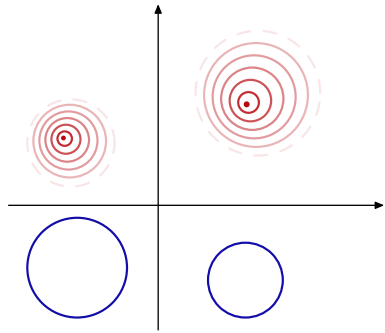
$$S_j(z,x,\vec{w} \mid G) = \sum_{\gamma \in G} \left(\frac{\mathrm{d}z}{z-\gamma x} - \frac{\mathrm{d}z}{z-\gamma P_j} \right) W(\gamma)$$

Similarly can split off one direction:

$$\begin{aligned} S_j(z,x,\vec{w} \mid G) &= \sum_{\tilde{\gamma} \in G_j \backslash G} \sum_{n \in \mathbb{Z}} W\left(\tilde{\gamma}^{-1} \gamma_j^{-n}\right) \frac{\mathrm{d}\left(\gamma_j^n \tilde{\gamma} z\right)}{\gamma_j^n \tilde{\gamma} z - x} \frac{P_1 - x}{P_1 - \gamma_j^n \tilde{\gamma} z} \\ &= \sum_{\tilde{\gamma} \in G_j \backslash G} W\left(\tilde{\gamma}^{-1}\right) S\left(\tilde{\gamma} z, x, w_j \mid G_j\right), \end{aligned}$$

$S\left(\tilde{\gamma} z, x, w_j \mid G_j\right)$ is (essentially) the Kronecker function from genus one!

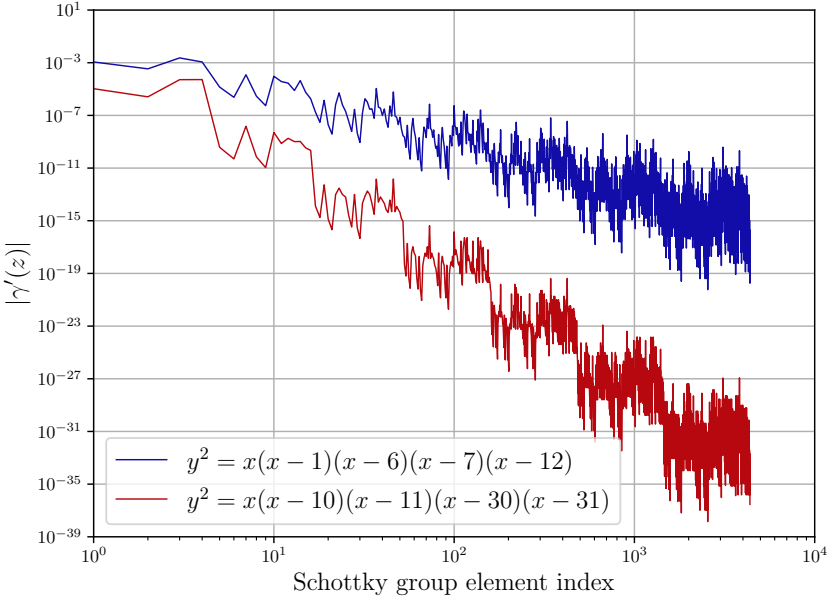
Use degeneration of the Schottky group to find



$$\omega_{i_1\cdots i_s j}(z,x \mid G) = -\frac{1}{2\pi i} \sum_{\gamma\in G/G_j} \sum_{k=0}^{\delta_{ji_s} n_s} C\left(b_{i_1}^{n_1}\cdots b_{i_s}^{n_s-k},\gamma\right) s^{(k)}\left(\gamma^{-1}z,x \mid G_j\right).$$

$\rightarrow$ can express higher kernels in terms of combinatorical Schottky sum of genus-one kernels $s^{(k)}$.

$\rightarrow$ can evaluate (and integrate) higher-genus polylogarithms explicitly.


10^{1}
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$|\gamma'(z)|$
10^{0}
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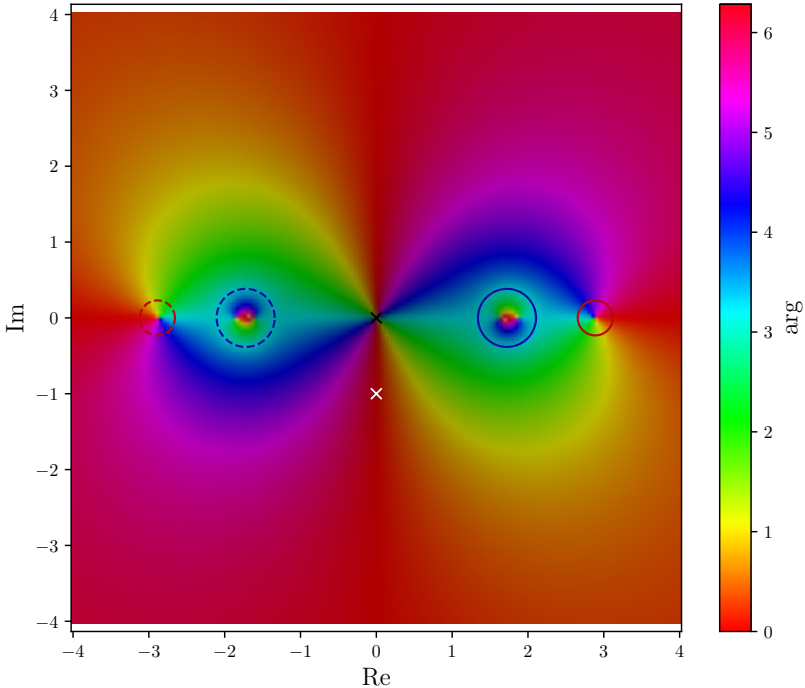
Schottky group element index
$y^2 = x(x-1)(x-6)(x-7)(x-12)$
$y^2 = x(x-10)(x-11)(x-30)(x-31)$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Kernel identities from Enriquez' generating function

with Konstantin Baune, Egor Im,
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First component form $K_{(\bullet)j}$:

$$K_{(b)j}(z,x) = \sum_{r\geq 0} \sum_{i_1,\ldots,i_r=1}^{h} \omega_{i_1\ldots i_r j}(z,x) b_{i_1}\ldots b_{i_r}, \quad j\in\{1,\ldots,h\}$$

(non-commutative) letters b_i generate a free Lie-algebra $\mathfrak{b}$

Properties:

$$K_{(b)j}(\gamma_k z, x) = e^{b_k} K_{(b)j}(z,x)$$
$$\mathrm{Res}_{z=x}\Big(K_{(b)j}(z,x)\Big) = -b_j$$

Quasi-periodicity in the auxiliary variable x:

$$K_{(b)j}(z,\gamma_k x) = K_{(b)j}(z,x) + K_{(b)k}(z,x)\frac{e^{-b_k}-1}{b_k} b_j.$$

Second component form $K_{(\bullet)j(\bullet)k}$:

$$K_{(b)j(\tilde{b})k}(z,x) = \sum_{r,s\geq 0} \sum_{i_1,\ldots,i_r=1}^{h} \sum_{p_1,\ldots,p_s=1}^{h} \omega_{i_1\ldots i_r j p_1 \ldots p_s k}(z,x) b_{i_1}\ldots b_{i_r} \tilde{b}_{p_1}\ldots\tilde{b}_{p_s}$$

$$j,k \in \{1,\ldots,h\}$$

$\tilde{\mathfrak{b}}$: second copy of the free Lie algebra, generated by generators $\tilde{b}_i$, with

$$\left[b_j, \tilde{b}_k\right] = 0\,.$$

$K_{(\bullet)j(\bullet)k}$ is valued in the universal enveloping algebra $U(\mathfrak{b}\oplus\tilde{\mathfrak{b}})$.

Properties:

$$K_{(b)j(\tilde{b})k}(\gamma_a z, x) = e^{b_a} K_{(b)j(\tilde{b})k}(z,x) + \delta_{ja}\frac{e^{\tilde{b}_a}-e^{b_a}}{\tilde{b}_a - b_a} K_{(\tilde{b})k}(z,x),$$

$$K_{(b)j(0)k}(z,\gamma_a x) = K_{(b)j(0)k}(z,x) + \delta_{jk} K_{(b)a}(z,x)\frac{e^{-b_a}-1}{b_a}$$

and

$$\mathrm{Res}_{z=x}\left(K_{(b)j(\tilde{b})k}(z,x)\right) = -\delta_{jk}.$$

Explore identities for the component forms...

... with two alphabets:

$$K_{(b+\tilde{b})j}(z,x) = \sum_{r,s\geq 0} \sum_{\substack{i_1,\ldots,i_r=1 \\ p_1,\ldots,p_s=1}}^{h} \omega_{(i_1\ldots i_r \sqcup p_1\ldots p_s)j}(z,x)\, b_{i_1}\ldots b_{i_r}\tilde{b}_{p_1}\ldots\tilde{b}_{p_s}.$$

$$K_{(b+\tilde{b})j(b)k}(z,x) = \sum_{s,t\geq 0} \sum_{r=0}^{t} \sum_{\substack{i_1,\ldots,i_t=1 \\ p_1,\ldots,p_s=1}}^{h} \omega_{(i_1\ldots i_r \sqcup p_1\ldots p_s)ji_{r+1}\ldots i_t k}(z,x) b_{i_1}\ldots b_{i_t}\tilde{b}_{p_1}\ldots\tilde{b}_{p_s}$$

for example:

$$K_{(b+\tilde{b})j(0)k}(\gamma_a z,x) = e^{b_a+\tilde{b}_a} K_{(b+\tilde{b})j(0)k}(z,x) + \delta_{ja}\frac{e^{b_a+\tilde{b}_a}-1}{b_a+\tilde{b}_a}\omega_k(z).$$

Explore identities for the component forms...

... with reversed words:

$$K_{(-b)j}(z,x) = \sum_{r\geq 0} \sum_{i_1,\ldots,i_r=1}^{h} (-1)^r \omega_{i_1\ldots i_r j}(z,x) b_{i_r}\ldots b_{i_1}$$

$$K_{(-b)j(\tilde{b})k}(z,x) = \sum_{r,s\geq 0} \sum_{\substack{i_1,\ldots,i_r=1\\ p_1,\ldots,p_s=1}}^{h} (-1)^r \omega_{i_1\ldots i_r j p_1\ldots p_s k}(z,x) b_{i_r}\ldots b_{i_1}\tilde{b}_{p_1}\ldots\tilde{b}_{p_s}$$

leading to

$$K_{(-b)j}(\gamma_a z,x) = K_{(-b)j}(z,x)e^{-b_a}$$

$$K_{(-b)j(\tilde{b})k}(\gamma_a z,x) = K_{(-b)j(\tilde{b})k}(z,x)e^{-b_a} + \delta_{aj}\frac{e^{\tilde{b}_a}-e^{-b_a}}{\tilde{b}_a+b_a}K_{(\tilde{b})k}(z,x).$$

Reversal of words leads to monodromy factors applied from the right instead of from the left.

Explore identities for the component forms...

... including contractions:

$$\sum_{j=1}^{h} K_{(b+\tilde{b})j(b)k}(z,x)\tilde{b}_j = K_{(b+\tilde{b})k}(z,x) - K_{(b)k}(z,x),$$

$$\sum_{j=1}^{h} \tilde{b}_j K_{(b)j(b+\tilde{b})k}(z,x) = K_{(b+\tilde{b})k}(z,x) - K_{(b)k}(z,x)$$

$$\sum_{j=1}^{h} \left(b_j K_{(\tilde{b})j(b)k}(z,x) - K_{(\tilde{b})j(b)k}(z,x)\tilde{b}_j \right) = K_{(b)k}(z,x) - K_{(\tilde{b})k}(z,x).$$

... from Enriquez' kernel identities:

[Enriquez]

$$j \neq k \Longrightarrow \omega_{i_1 \cdots i_r jk}(z,x) \equiv \omega_{i_1 \cdots i_r jk}(z),$$
$$\omega_{i_1 \cdots i_r jj}(z,x) - \omega_{i_1 \cdots i_r kk}(z,x) \equiv \phi_{i_1 \cdots i_r, j,k}(z),$$

$$j \neq k \Longrightarrow K_{(b)j(0)k}(z,x) = K_{(b)j(0)k}(z,y)$$
$$K_{(b)j(0)j}(z,x) - K_{(b)k(0)k}(z,x) = K_{(b)j(0)j}(z,y) - K_{(b)k(0)k}(z,y)$$
$$\delta_{kl}\left(K_{(\tilde{b})l}(y,z) - K_{(\tilde{b})l}(y,x) \right) = \left(K_{(\tilde{b})k(0)l}(y,z) - K_{(\tilde{b})k(0)l}(y,x) \right) \tilde{b}_l.$$

The above identities can be combined to prove the following:

(Index-swapping) identities:

$$0 = \mathcal{S}_{kl}(z,y,x) = \sum_{j=1}^{h} K_{(b)j}(z,x)\Big(K_{(\tilde{b})j(0)k}(y,z) - K_{(\tilde{b})j(0)k}(y,x) - \delta_{jk}K_{(\tilde{b})l(0)l}(y,z) + \delta_{jk}K_{(\tilde{b})l(0)l}(y,x)\Big).$$

for arbitrary $k,l \in \{1,\ldots,h\}$.

$$0 = \mathcal{S}_{k}(z,y,x) = \sum_{j=1}^{h} \Bigg(K_{(\tilde{b})j}(y,z)K_{(b)j(b+\tilde{b})k}(z,x) - \Big[K_{(\tilde{b})j(0)k}(y,z) - K_{(\tilde{b})j(0)k}(y,x)\Big]\Big[K_{(b+\tilde{b})j}(z,x) - K_{(b)j}(z,x)\Big] - K_{(\tilde{b})j}(y,x)K_{(b)j(b+\tilde{b})k}(z,x)\Bigg)$$

Quadratic 3-point identity:

$$\begin{aligned}0 = \mathcal{F}_k(z,y,x) =& K_{(b)j}(z,x)K_{(\tilde{b})k}(y,x)a_j\tilde{b}_i\tilde{a}_i + K_{(b)k}(z,x)K_{(\tilde{b})j}(y,x)\tilde{a}_j b_i a_i \\ &- K_{(b)j}(z,y)\left[K_{(\tilde{b})k}(y,x)a_j + K_{(\tilde{b})j(b+\tilde{b})k}(y,x)b_i a_i\right]\tilde{b}_{i'}\tilde{a}_{i'} \\ &- K_{(\tilde{b})j}(y,z)\left[K_{(b)k}(z,x)\tilde{a}_j + K_{(b)j(b+\tilde{b})k}(z,x)\tilde{b}_i\tilde{a}_i\right]b_{i'}a_{i'}\end{aligned}$$

(summation over j, i, i', where appropriate).

Remarks:

- reduces to the Fay identities for the Kronecker generating function for meromorphic kernels at genus one.
- can be rewritten to yield all identities for meromorphic kernels from [D'Hoker Schlotterer], thus proving their conjectures for Enriquez' kernels.
- Can prove: the class of identities above exhausts all quadratic three-point identities.

Example:

$$\begin{aligned}\tilde{\Gamma}\left(\begin{smallmatrix}(11),&(11)\\z_0,&z\end{smallmatrix};z\middle|G\right) = &-\int_{t_0=z_0}^{z}\omega_{111}(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_1(t_1,t_0)\\&-\int_{t_0=z_0}^{z}\omega_1(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_{111}(t_1,t_0)\\&-\int_{t_0=z_0}^{z}\omega_{211}(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_2(t_1,t_0)\\&-\int_{t_0=z_0}^{z}\omega_{21}(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_{12}(t_1,t_0)\\&-\int_{t_0=z_0}^{z}\omega_2(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_{121}(t_1,t_0)\\&-\tilde{\Gamma}\left(\begin{smallmatrix}(2),&(211)\\z_0,&z_0\end{smallmatrix};z\middle|G\right)-\tilde{\Gamma}\left(\begin{smallmatrix}(1),&(111)\\z_0,&z_0\end{smallmatrix};z\middle|G\right)\\&+\frac{1}{24}.\end{aligned}$$

Example:

$$\begin{aligned}\tilde{\Gamma}\left(\begin{smallmatrix}(11),&(11)\\z_0,&z\end{smallmatrix};z\middle|G\right) = &-\int_{t_0=z_0}^{z}\omega_{111}(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_1(t_1,t_0)\\ &-\int_{t_0=z_0}^{z}\omega_1(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_{111}(t_1,t_0)\\ &-\int_{t_0=z_0}^{z}\omega_{211}(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_2(t_1,t_0)\\ &-\int_{t_0=z_0}^{z}\omega_{21}(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_{12}(t_1,t_0)\\ &-\int_{t_0=z_0}^{z}\omega_2(t_0,z_0)\int_{t_1=z_0}^{t_0}\omega_{121}(t_1,t_0)\\ &-\tilde{\Gamma}\left(\begin{smallmatrix}(2),&(211)\\z_0,&z_0\end{smallmatrix};z\middle|G\right)-\tilde{\Gamma}\left(\begin{smallmatrix}(1),&(111)\\z_0,&z_0\end{smallmatrix};z\middle|G\right)\\ &+\frac{\zeta_2}{-(2\pi i)^2}.\end{aligned}$$

Summary

- Kronecker function from genus one can be extended to higher genera. All analytic constraints are met. The meromorphic result is equivalent to Enriquez' generating function.
- Quasiperiodicities along $\mathfrak{B}$-cycles are noncommuting, needs to be adressed algebraically. Algebraic identities dictate quadratic 3-point identities.
  Space of quadratic identities exhausted.
- In general, Riemannian manifolds correspond to submanifolds in the Jacobian $\rightarrow$ calculationally extremely unwieldy. Circumvented by use of Schottky uniformization and Schottky-expansion into genus-one kernels.
- Enriquez' kernels can be evaluated numerically in the Schottky uniformization.
  $\rightarrow$ Evaluate higher-genus polylogarithms numerically.
  $\rightarrow$ Check functional relations between polylogarithms numerically.
- Convergence accelerators (Ginac, PARI) through modular transformations for ell. polylogs, MZVs effectively use symmetries of the Schottky parametrization. Better/faster convergence through clever (higher) modular transformations paired with power of automorphic forms/Schottky sums?

Further remarks & open questions

- Kernel identities exhausted for the (meromorphic) language from [Enriquez][Enriquez Zerbini]
  Translate to the fully periodic language of [D'Hoker, Hidding Schlotterer]?

- Translate results into algebraic/hyperelliptic curve language?
  Abelian differentials known $\rightarrow$ particle physics applications.

  Connect to calculations of genus-drop integrals (similar) [Marzucca, McLeod, Page Poegel, Weinzierl]

- Generalization of the Eisenstein representation of Kronecker form at genus one?
  Requires understanding of higher uniformizations ($\mathbb{H}/$(Fuchsian group)).
  Need to explore Siegel modular forms for congruence subgroups.

- Coaction?
  Algebraically involved, at least. But probably straightforward....

- Define and explore higher multiple zeta values and their relations.
  (Tools are there. First results as well...)

- Relation to *Fay's trisecant equation* and geometric interpretation of the kernel identities completely unsettled beyond genus one.

Thanks!