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Lessons from ϵ-factorizing elliptic (and beyond) differential equations

Elliptics and Beyond 2024

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based on
2401.14307 with **Mathieu Giroux**, **Andrzej Pokraka**, **Franziska Porkert**
and work in progress with **Sara Maggio**

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Differential Equations [Kotikov, 1991]

- For a given **Feynman Integral** we can generate **differential equations (deq)s** algorithmically [Laporta, hep-ph/0102033]

- The deq involves a set of **Master Integrals (MIs)**, **I**:

$$\mathrm{d}\mathbf{I} = \underbrace{\mathcal{A}(D, p^2, m_i^2)}_{\text{rational fcts.}}\,\mathbf{I};\quad D = 2 - 2\epsilon$$

- We try to find the "ϵ-**form**" of this equation through a gauge transformation $\mathbf{U}\cdot\mathbf{I} := \mathbf{J}$ s.t. [Henn, 1304.1806]:

$$\mathrm{d}\mathbf{J} = \epsilon\,\underbrace{\tilde{\mathcal{A}}(p^2, m_i^2)}_{\epsilon\text{ independent}}\,\mathbf{J} \quad\rightarrow\quad \mathbf{J} = \mathbb{P}\exp\left[\epsilon\int_\gamma \tilde{\mathcal{A}}\right]\mathbf{J}_0$$

- **U** can introduce **periods** of **elliptic curves** (or worse)

Differential Equations [Kotikov, 1991]

- The differential equation has the following shape:

$$\mathbf{d}\begin{pmatrix}\vec{I}_1\\ \vec{I}_2\end{pmatrix} = \begin{pmatrix}\mathcal{A}_{1,1} & 0\\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2}\end{pmatrix} \cdot \begin{pmatrix}\vec{I}_1\\ \vec{I}_2\end{pmatrix}$$

- $\vec{I}_2$ corresponds to **sector** of target integral and $\vec{I}_1$ all its subtopologies.

- We call $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,2}$ the **maximal cut connections** of $\vec{I}_1$ and $\vec{I}_2$ respectively and $\mathcal{A}_{2,1}$ the **inhomogeneous connection**.

- To ϵ-factorize a full connection we proceed in two steps:

  - **First** the maximal cut connections $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,2}$

  - **Second** the inhomogeneous connection $\mathcal{A}_{2,1}$

ϵ-factorization of the maximal cut

- **Easiest Case:** $\mathcal{A}_{2,2}$ is a **scalar** and linear in ϵ (we set $x_i = m_i^2/p^2$)

  e.g.: Eyeball **[Giroux, Pokraka, Porkert, YS, 2401.14307]**

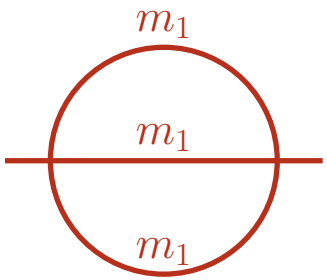
$$\mathrm{d}I_2 = \underbrace{\left(\mathcal{A}_{2,2}^{(0)} + \epsilon\mathcal{A}_{2,2}^{(1)}\right) I_2}_{\text{max cut}} + \underbrace{\mathcal{A}_{2,1}\vec{I}_1}_{\text{inhom.}}$$

- Master integral $J_2 = I_2/I_2^{\text{cut}}$ ϵ-factorizes the maximal cut connection.

$$\mathrm{d}I_2^{\text{cut}} = \mathcal{A}_{2,2}^{(0)} I_2^{\text{cut}} \quad \text{in this case} \quad I_2^{\text{cut}} = \frac{1}{\sqrt{P_4(x_4)}}$$

ϵ-factorization of the maximal cut

- **Harder Case:** $\mathcal{A}_{2,2}$ is a **matrix** and quadratic in ϵ



$x = m_1^2/p^2$

e.g.: Sunrise [**Müller-Stach, Weinzierl, Zayadeh, 1112.4360**]

$$\mathbf{d}\begin{pmatrix} I_2 \\ \partial_x I_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}x \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \end{pmatrix} \cdot \begin{pmatrix} I_2 \\ \partial_x I_2 \end{pmatrix}$$

- More useful language: the **Picard-Fuchs** operator:

$$\partial_x^2 I_2 = (\alpha(x) + \epsilon\beta(x))\,\partial_x I_2 + \Big(A(x) + \epsilon B(x) + \epsilon^2 C(x)\Big)\, I_2$$

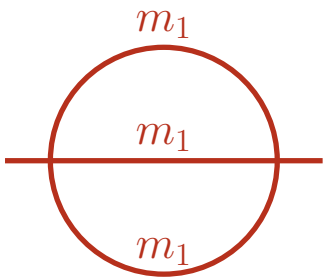
can be read off

- ϵ-form can be reached when $\partial_x^2 I_2 = \beta(x)\epsilon\partial_x I_2 + C(x)\epsilon^2 I_2$

ϵ-factorization of the maximal cut

- **Harder Case:** $\mathcal{A}_{2,2}$ is a **matrix** and quadratic in ϵ

  e.g.: Sunrise **[Müller-Stach, Weinzierl, Zayadeh, 1112.4360]**



$$\mathbf{d}\begin{pmatrix} I_2 \\ \partial_x I_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}x \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \end{pmatrix} \cdot \begin{pmatrix} I_2 \\ \partial_x I_2 \end{pmatrix}$$

- More useful language: the **Picard-Fuchs** operator:

$$\partial_x^2 I_2 = \left(\alpha(x) + \epsilon\beta(x)\right)\partial_x I_2 + \left(A(x) + \epsilon B(x) + \epsilon^2 C(x)\right) I_2$$

- The desired ϵ-form is reached through:

$$\partial_x \underbrace{\left(\left(\frac{\mathrm{Jac}}{\epsilon}\partial_x + F(x)\right)\frac{I_2}{\psi_1}\right)}_{J_2} = \epsilon\left\{\gamma(x)\underbrace{\left(\frac{\mathrm{Jac}}{\epsilon}\partial_x + F(x)\right)\frac{I_2}{\psi_1}}_{J_2} + D(x)\underbrace{\frac{I_2}{\psi_1}}_{J_1}\right\}$$

ϵ-factorization of the maximal cut

$$\partial_x \underbrace{\left(\left(\frac{\text{Jac}}{\epsilon}\partial_x + F(x)\right)\frac{I_2}{\psi_1}\right)}_{J_2} = \epsilon\left\{\gamma(x)\underbrace{\left(\frac{\text{Jac}}{\epsilon}\partial_x + F(x)\right)\frac{I_2}{\psi_1}}_{J_2} + D(x)\underbrace{\frac{I_2}{\psi_1}}_{J_1}\right\}$$

- Where ψ_1 solves the deq at $\epsilon = 0$, it is the **Period** of elliptic curve.

$$\partial_x^2\psi_j = \alpha(x)\partial_x\psi_j + A(x)\psi_j\ , \quad j = 1,2\,, \qquad \tau = \frac{\psi_2}{\psi_1}$$

- Desired **Jacobian** determined by $\text{Jac} = \dfrac{\partial x}{\partial \tau}$

- $F(x)$ tells how **MIs** couple to each other, solved by **linear** deq.

- ϵ-factorize by changing to **MIs**: $\vec{J} = \left(\dfrac{I_2}{\psi_1}, \left(\dfrac{\partial_\tau}{\epsilon} + F\right)\dfrac{I_2}{\psi_1}\right)^{\text{T}}$

- Generalisation for **any loop**, but only **one scale**.

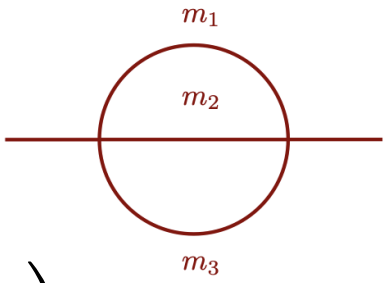
[Pögel, Wang, Weinzierl, 2212.08908]

ϵ-factorization of the maximal cut

- **Generalising to multiple scales:** [Maggio, YS, WIP]

Unequal Mass Sunrise $(x_i = m_i^2/p^2)$

[Bogner, Müller-Stach, Weinzierl, 1907.01251]



$$\mathbf{d}\begin{pmatrix} I_2 \\ \partial_{x_1} I_2 \\ \partial_{x_2} I_2 \\ \partial_{x_3} I_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}x_1 & \mathrm{d}x_2 & \mathrm{d}x_3 \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) \end{pmatrix} \begin{pmatrix} I_2 \\ \partial_{x_1} I_2 \\ \partial_{x_2} I_2 \\ \partial_{x_3} I_2 \end{pmatrix}$$

- We find two possibilities to ϵ-factorise: the **democratic** and the **decoupling** approach.

ϵ-factorization of the maximal cut

- **The democratic approach**:

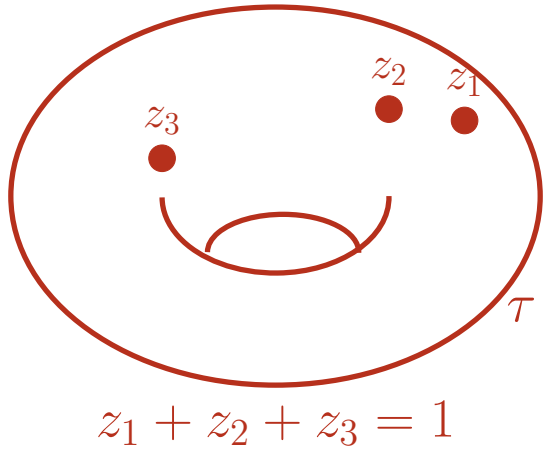
$$\vec{J} = \left(\frac{I_2}{\psi_1}, \quad \left(\frac{\partial_\tau}{\epsilon} + F_1\right)\frac{I_2}{\psi_1}, \quad \left(\frac{\partial_{z_1}}{\epsilon} + F_2\right)\frac{I_2}{\psi_1}, \quad \left(\frac{\partial_{z_2}}{\epsilon} + F_3\right)\frac{I_2}{\psi_1}\right)^{\mathrm{T}}$$

- where τ, z_1 and z_2 are degrees of freedom in the **Moduli space** $\mathcal{M}_{1,3}$

- The initial **Picard-Fuchs** at $\epsilon = 0$ is solved by $\psi_1 z_1$ and $\psi_1 z_2$

- Non-diagonal **Jacobian**:

$$\begin{pmatrix} \partial_\tau \\ \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} = \begin{pmatrix} \partial_\tau x_1 & \partial_\tau x_2 & \partial_\tau x_3 \\ \partial_{z_1} x_1 & \partial_{z_1} x_2 & \partial_{z_1} x_3 \\ \partial_{z_2} x_1 & \partial_{z_2} x_2 & \partial_{z_2} x_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$



ϵ-factorization of the maximal cut

- **The decoupling approach**: reduction algorithms also give $I^{(a)}_{\text{dec}}$, $I^{(b)}_{\text{dec}}$

  Start from $\vec{I}_2 = \left(I_2, I^{(a)}_{\text{dec}}, I^{(b)}_{\text{dec}}, \partial_{x_1} I_2\right)^T$

- $I^{(a)}_{\text{dec}}$ and $I^{(b)}_{\text{dec}}$ have **irreducible scalar products** as numerators

  (namely $I_2 = I_{1,1,1,0,0}$, $I^{(a)}_{\text{dec}} = I_{1,1,1,-1,0}$ and $I^{(b)}_{\text{dec}} = I_{1,1,1,0,-1}$)

$$\vec{J} = \left(\frac{I_2}{\psi_1}, \underbrace{I^{(a)}_{\text{dec}} + F_1\frac{I_2}{\psi_1}, \; I^{(b)}_{\text{dec}} + F_2\frac{I_2}{\psi_1}}_{\text{only couple to } I_2/\psi_1}, \; \left(\frac{\partial_\tau}{\epsilon} + F_3\right)\frac{I_2}{\psi_1} + F_4 I^{(a)}_{\text{dec}} + F_5 I^{(b)}_{\text{dec}}\right)^T$$

- The initial **Picard-Fuchs** at $\epsilon = 0$ still eliminates ψ_1 but **no longer** eliminates $\psi_1 z_1$ and $\psi_1 z_2$ and **Jacobian** becomes $\partial_\tau = (\partial_\tau x_1)\, \partial_{x_1}$

ϵ-factorization of the maximal cut

- **New Result:** The Three-Loop Banana: need **both** methods

- We find four **Periods** of underlying **K3-surface**, [Duhr, Maggio, WIP]

  $\psi_0, \psi_1^{(a)}, \psi_1^{(b)}, \psi_2$. After normalising: $\tau_1 = \psi_1^{(a)}/\psi_0$ and $\tau_2 = \psi_1^{(b)}/\psi_0$

$$J_1 = \frac{I_2}{\psi_0}$$

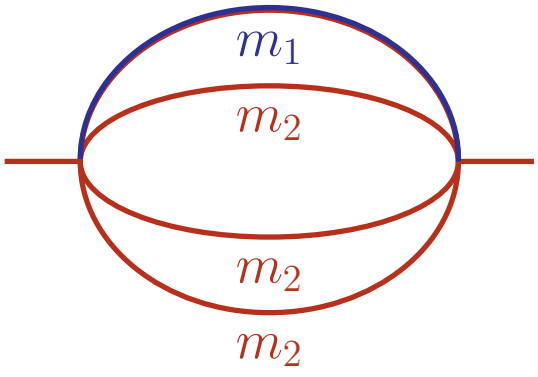
$$J_2 = I_{\text{dec}} + F_1 J_1$$

$$J_3 = \left(\frac{\partial_{\tau_1}}{\epsilon} + F_2\right) J_1 + F_3 J_2$$

$$J_4 = \left(\frac{\partial_{\tau_2}}{\epsilon} + F_4\right) J_1 + F_5 J_2$$

$$J_5 = \left(\frac{\partial_{\tau_1}}{\epsilon} + F_6\right) J_3 + F_7 J_1 + F_8 J_2 + F_9 J_4$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 2 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$



ϵ-factorization of the maximal cut

- **New Result:** The Three-Loop Banana, different mass configuration.

- Again **four Periods** and construct $\tau_1 = \psi_1^{(a)}/\psi_0$ and $\tau_2 = \psi_1^{(b)}/\psi_0$

$$J_1 = \frac{I_2}{\psi_0}$$

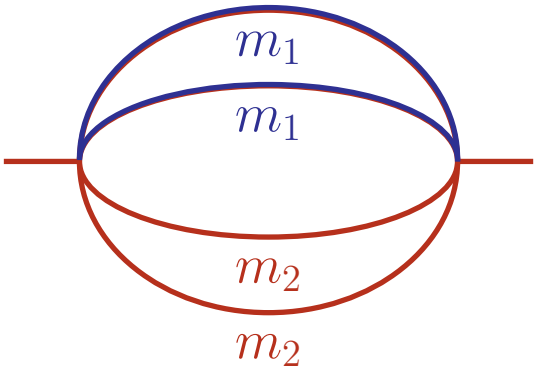
$$J_2 = I^{(a)}_{\text{dec}} + F_1 J_1$$

$$J_3 = I^{(b)}_{\text{dec}} + F_2 J_1 \quad \Leftarrow \text{new } \mathbf{MI}$$

$$J_4 = \left(\frac{\partial_{\tau_1}}{\epsilon} + F_2\right) J_1 + F_3 J_2 + F_4 J_3$$

$$J_5 = \left(\frac{\partial_{\tau_2}}{\epsilon} + F_5\right) J_1 + F_6 J_2 + F_7 J_3$$

$$J_6 = \left(\frac{\partial_{\tau_1}}{\epsilon} + F_8\right) J_4 + F_9 J_1 + F_{10} J_2 + F_{11} J_3 + F_{12} J_5$$



ϵ-factorization of the inhomogeneities

$$\mathbf{d}\begin{pmatrix}\vec{I}_1(\vec{X}_1)\\ \vec{I}_2(\vec{X}_1,\vec{X}_2)\end{pmatrix} = \begin{pmatrix}\overbrace{\epsilon\mathcal{A}_{1,1}(\vec{X}_1)}^{\text{done}} & 0\\ \underbrace{\mathcal{A}^{(0)}_{2,1}(\vec{X}_1,\vec{X}_2)+\epsilon\mathcal{A}^{(1)}_{2,1}(\vec{X}_1,\vec{X}_2)}_{\text{inhomogeneity}} & \underbrace{\epsilon\mathcal{A}_{2,2}(\vec{X}_1,\vec{X}_2)}_{\text{done}}\end{pmatrix}\cdot\begin{pmatrix}\vec{I}_1\\ \vec{I}_2\end{pmatrix}$$

- **Solved by**: $\vec{I}_2 \to \vec{I}_2 + F\vec{I}_1$ with $F \propto -\int \mathcal{A}^{(0)}_{2,1}$
- **Easiest Case:** $\mathcal{A}_{2,2}$ is **elliptic** and $\mathcal{A}_{1,1}$ is **polylogarithmic**: we already know: $(\vec{X}_1,\vec{X}_2) \to (\tau,\vec{z})$

  We express F with (elliptics)!

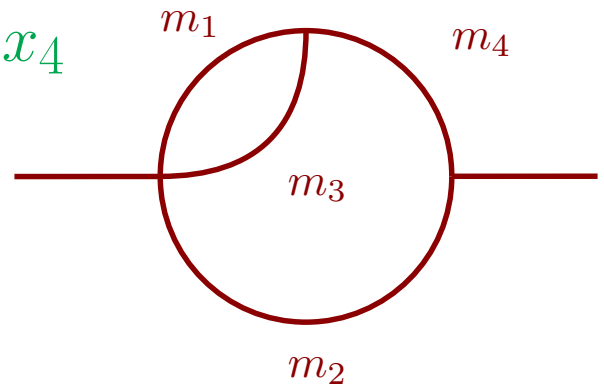
ϵ-factorization of the inhomogeneities

$$\mathbf{d}\begin{pmatrix}\vec{I}_1(\vec{X}_1)\\ \vec{I}_2(\vec{X}_1,\vec{X}_2)\end{pmatrix} = \begin{pmatrix}\epsilon\mathcal{A}_{1,1}(\vec{X}_1) & 0\\ \mathcal{A}^{(0)}_{2,1}(\vec{X}_1,\vec{X}_2)+\epsilon\mathcal{A}^{(1)}_{2,1}(\vec{X}_1,\vec{X}_2) & \epsilon\mathcal{A}_{2,2}(\vec{X}_1,\vec{X}_2)\end{pmatrix}\cdot\begin{pmatrix}\vec{I}_1\\ \vec{I}_2\end{pmatrix}$$

- **Solved by**: $\vec{I}_2 \to \vec{I}_2 + F\vec{I}_1$ with $F \propto -\int \mathcal{A}^{(0)}_{2,1}$
- **Harder Case:** $\mathcal{A}_{2,2}$ is **polylogarithmic** and $\mathcal{A}_{1,1}$ is **elliptic**: we already know $\vec{X}_1 \to$ (elliptics) but not how to map $\vec{X}_2$!
- **Eyeball**: **four** scales $\vec{X}_1 = (x_1, x_2, x_3)$ and $\vec{X}_2 = x_4$

  From $\mathcal{A}_{1,1}$ we know that $(x_1, x_2, x_3) \to (\tau, z_1, z_2)$

  What about x_4?

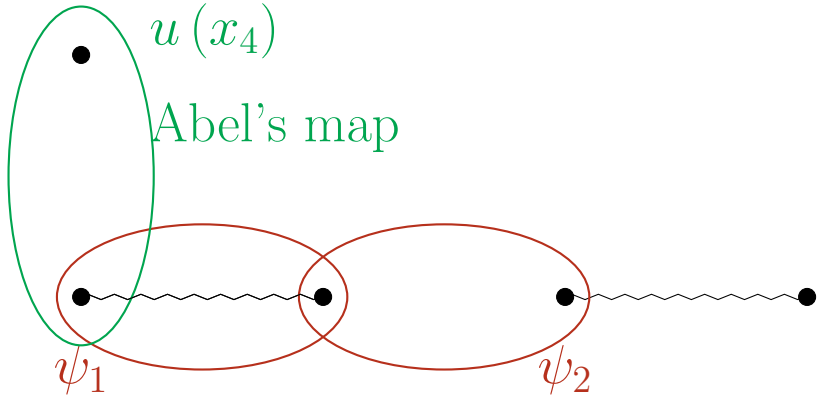


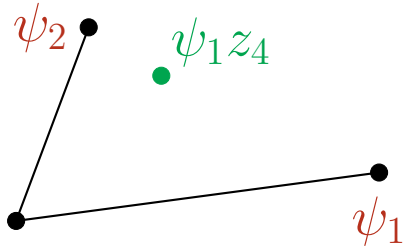
ϵ-factorization of the inhomohegeities

- We should find how x_4 maps to the torus! We find a natural candidate:

$$\int_0^{x_4} \mathrm{d}x_4' \, I_2^{\text{cut}} = \int_0^{x_4} \frac{\mathrm{d}x_4'}{\sqrt{P_4(x_4')}} = \psi_1 z_4$$

- This turns out to be **Abel's map** (inverse of $\wp(z)$) for our torus!





- We now have a map $(x_1, x_2, x_3, x_4) \to (\tau, z_1, z_2, z_4)$ and can express $F \propto -\int \mathcal{A}_{2,1}^{(0)}$ in terms of **elliptic** functions

Conclusion and Outlook

- We increased the arsenal of ϵ-factorizing methods.

  [Görges, Nega, Tancredi, Wagner, 2305.14090]

  We find that a **derivative Ansatz** helps to ϵ-factorise maximal-cuts:

  $\Rightarrow$ Need to understand decoupling **MIs** and correct **variables**.

  Integration of maximal cuts can give us **new variables**.

- **Next**: Look at integrals with more than two legs (e.g. double box).
- Is ϵ-form the **canonical** form? Study the pole structure.
- Expressing the result in terms of **(quasi)-modular-forms** for integrals beyond elliptics.

Thank you for your attention !