# Elliptic Feynman integrals in the local-in-time conservative binary dynamics at 4PM

[CD, Kälin, Liu, Porto, '24] 2403.04853

European Research Council
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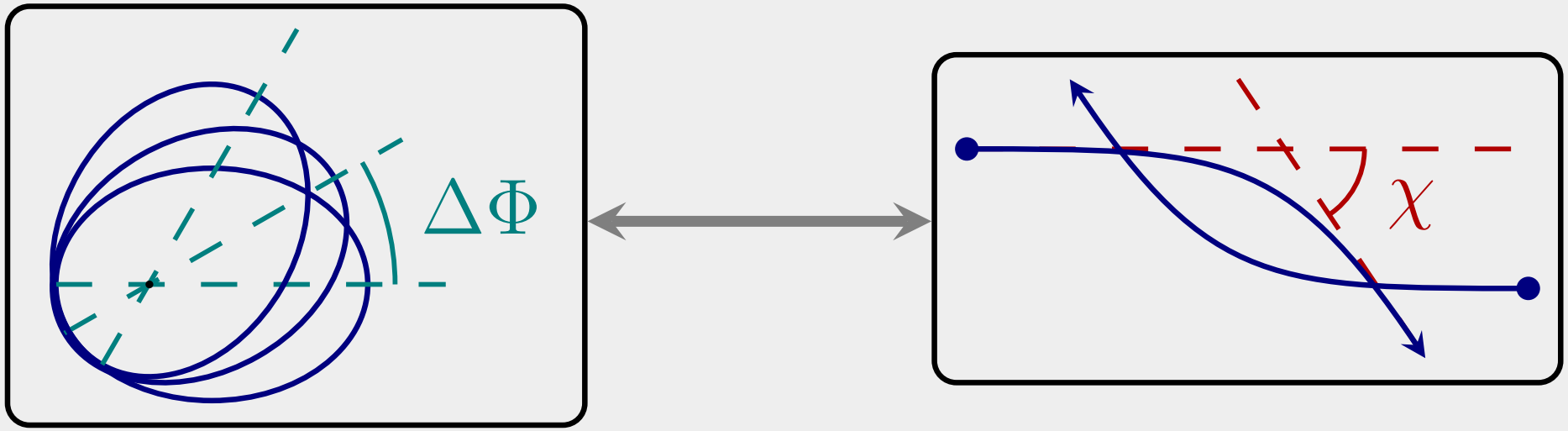
Christoph Dlapa

# Gravitational two-body problem: B2B

[Kälin, Porto, '20]

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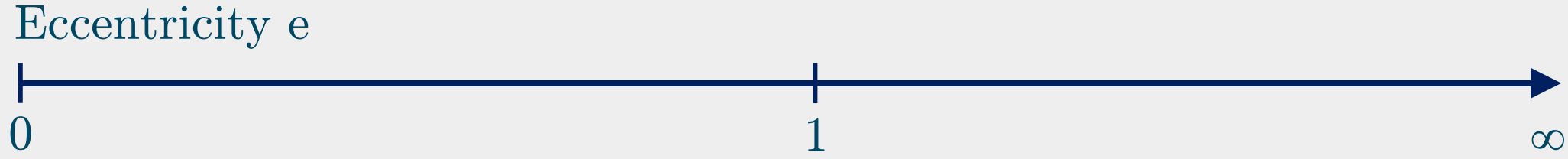
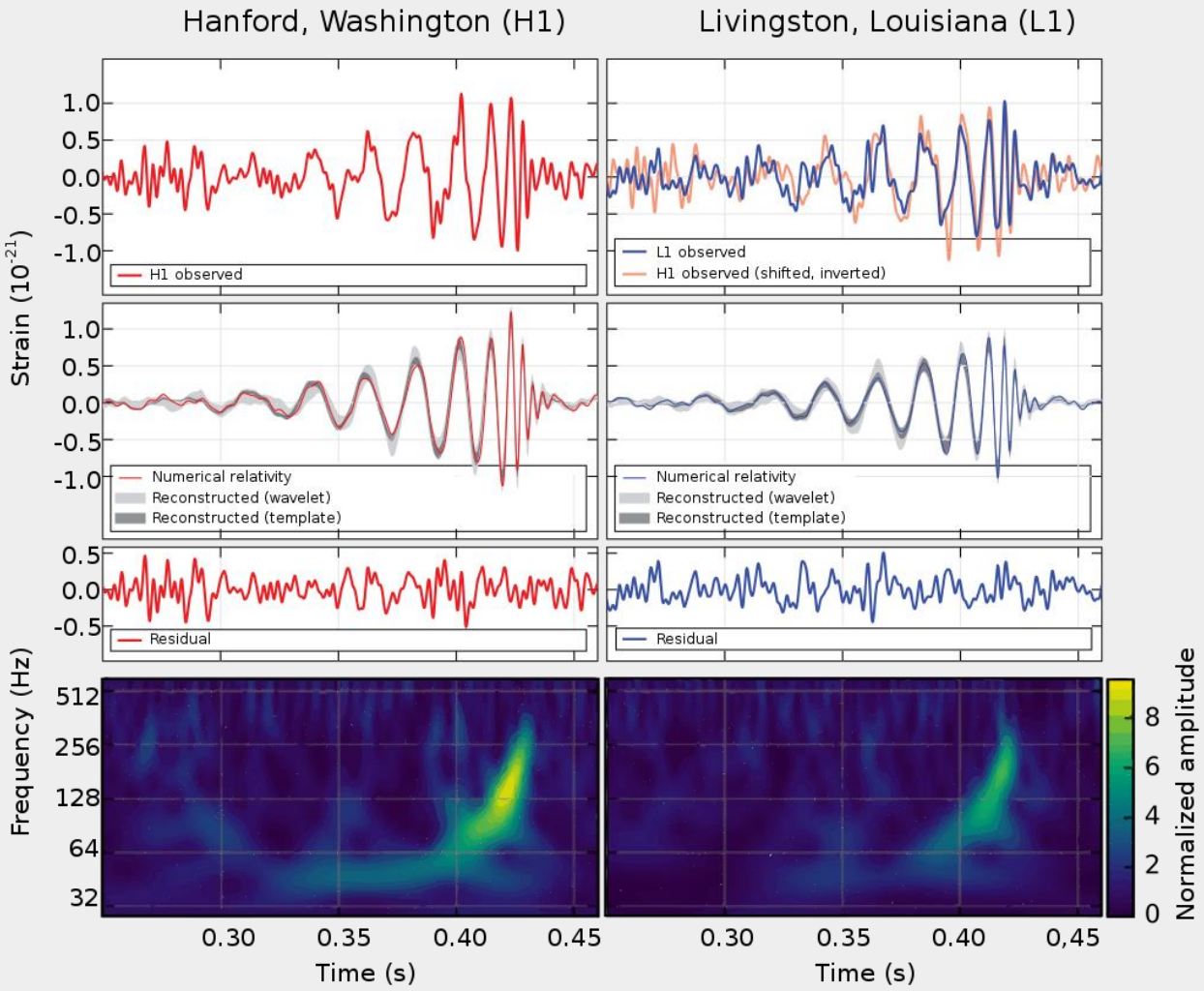
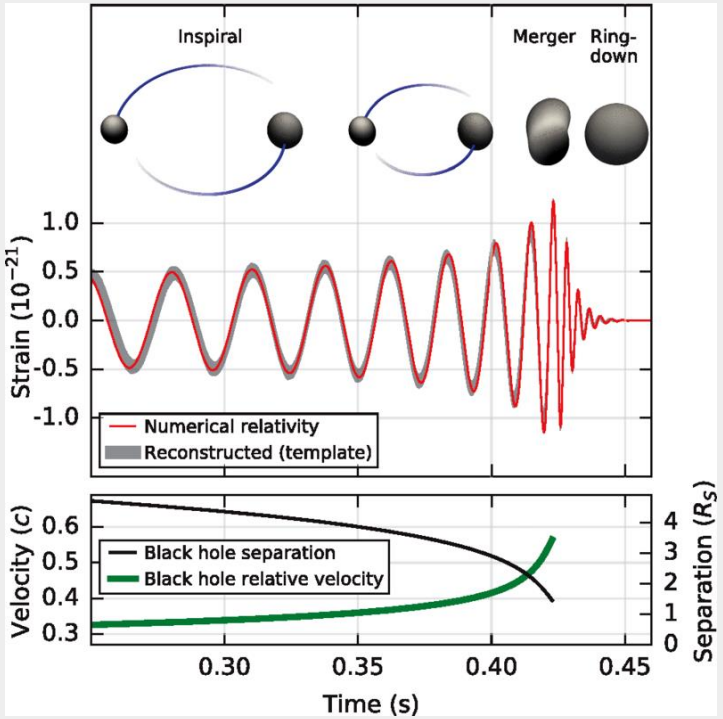
analytic continuation
in binding energy

$\mathcal{E} < 0$



# Waveforms




[LIGO]

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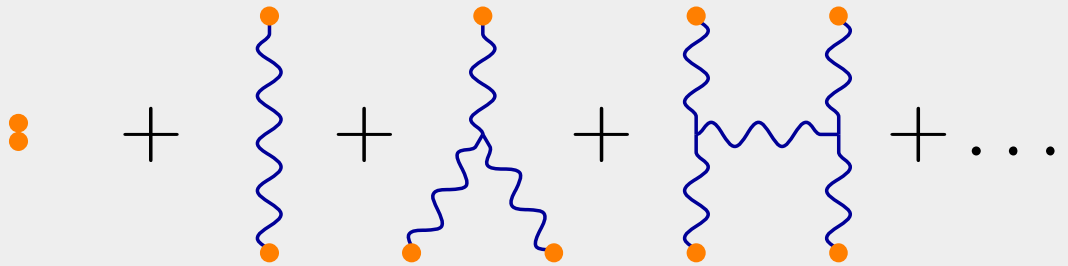
$S_{\rm eff} =$  $+ \ldots$

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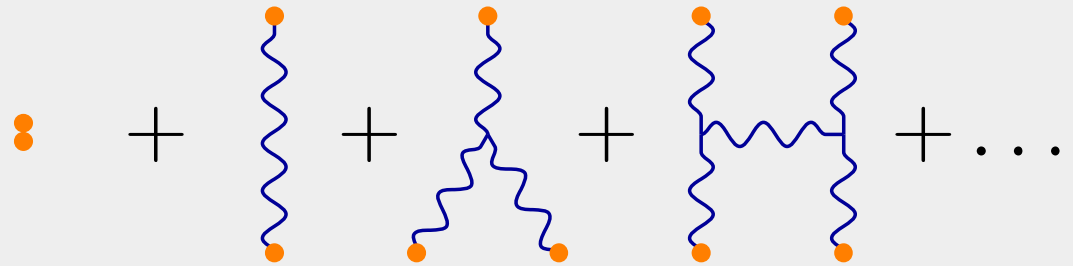
$S_{\rm eff} =$  $= -\sum_{a=1,2} \frac{m_a}{2} \int \mathrm{d}\tau \, \eta_{\mu\nu} v_a^\mu v_a^\nu$

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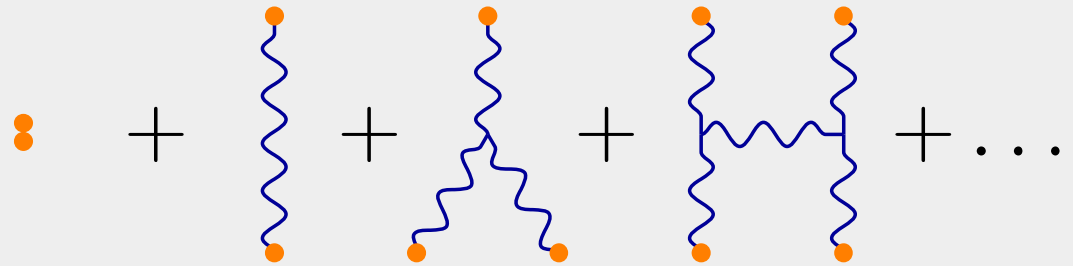
$S_{\rm eff} =$  $+ \dots$

$$= -\sum_{a=1,2} \frac{m_a}{2} \int \mathrm{d}\tau \, \eta_{\mu\nu} v_a^\mu v_a^\nu$$
$$+ G \frac{i}{4} \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \, v_1^\alpha v_1^\beta v_2^\mu v_2^\nu \int \mathrm{d}^4\ell \frac{P_{\alpha\beta\mu\nu}}{\ell^2} e^{i\ell\cdot(x_1 - x_2)}$$

# Classical equations of motion

$$S_{\text{eff}} = \sum_{n=0}^{\infty} G^n S_n = -\sum_{a=1,2} \frac{m_a}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \eta_{\mu\nu} v_a^\mu v_a^\nu + \dots, \qquad S_n = \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \mathcal{L}_n[x_1, x_2]$$

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observable/integrand

$$u_{\text{com}} \equiv \frac{m_1 u_1 + m_2 u_2}{E}, \qquad = (1,0,0,0)_{\text{com}}$$

# Loop integrals at 3PM

cut propagator

- PP propagators: $\int_{-\infty}^{\tau_1} \mathrm{d}\tilde{\tau}_1 e^{i\tilde{\tau}_1(\ell\cdot u_1)} = \frac{-i\, e^{i\tau_1(\ell\cdot u_1)}}{\ell\cdot u_1}, \quad \int_{-\infty}^{\infty} \mathrm{d}\tau_1 e^{i\tau_1(\ell\cdot u_1)} = 2\pi\,\delta(\ell\cdot u_1)$

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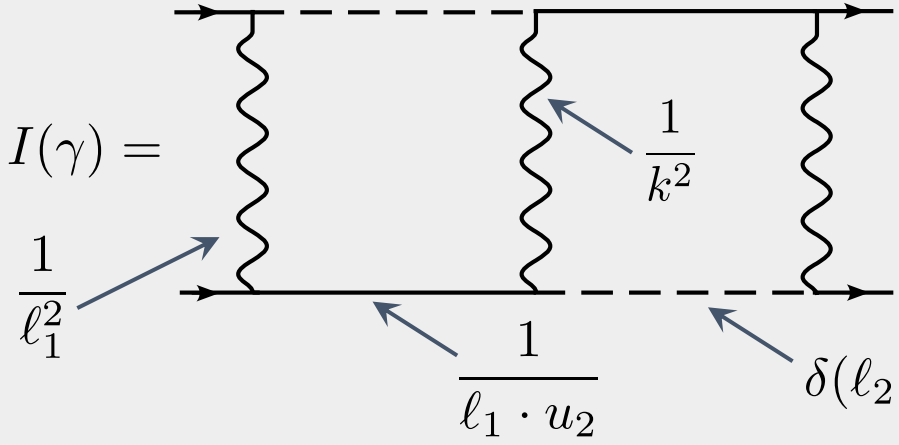
$D = 4 - 2\epsilon$

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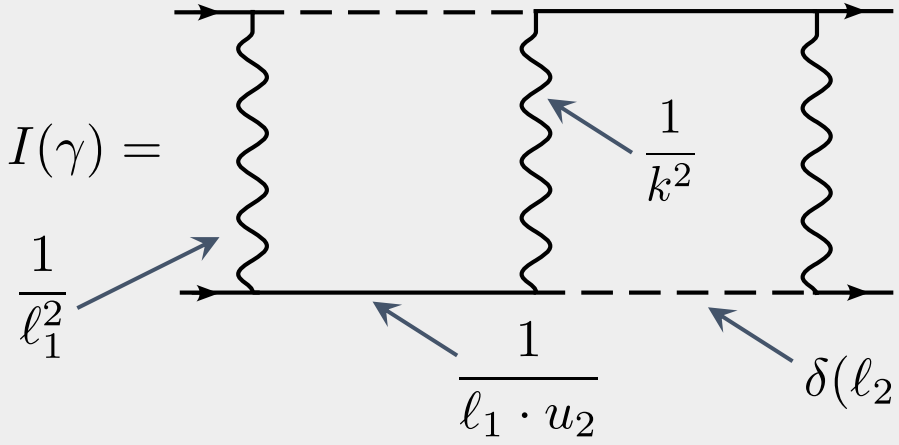


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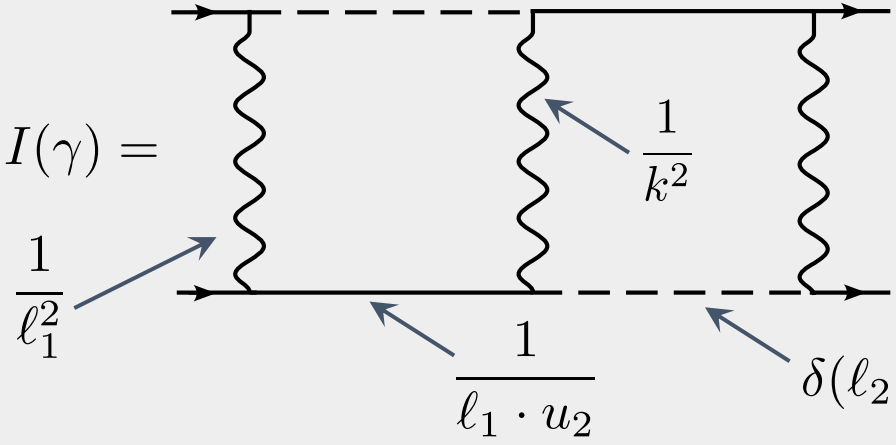


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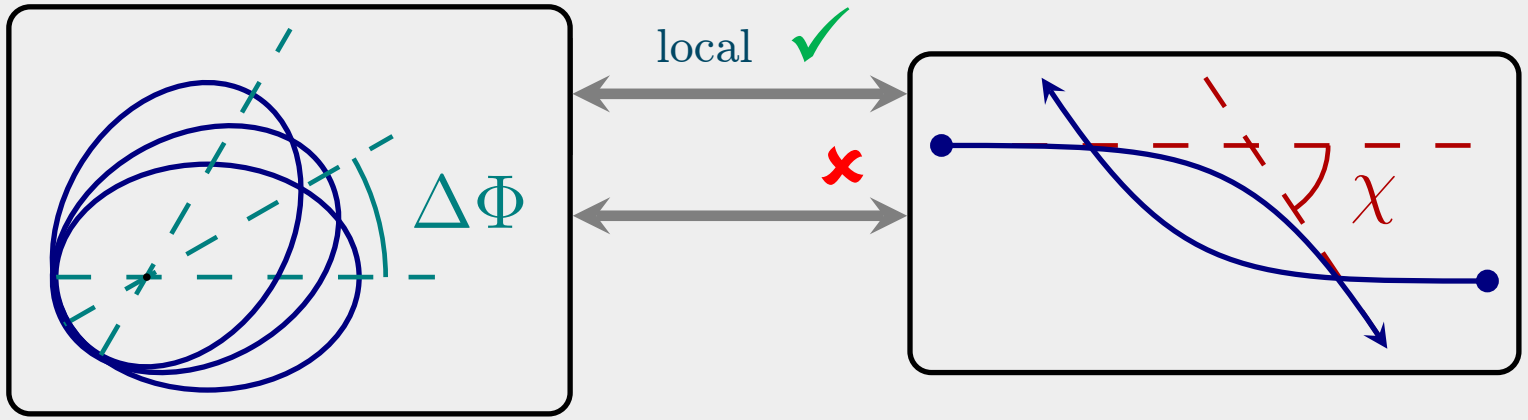
- Rationalize square-root: $\gamma = \frac{1}{2}\left(x + \frac{1}{x}\right), \qquad v \equiv \sqrt{\gamma^2-1} = \frac{1}{2}\left(\frac{1}{x} - x\right)$

# Nonlocal-in-time contributions




$\Delta p_1 \to \Delta E$

$\to \chi$

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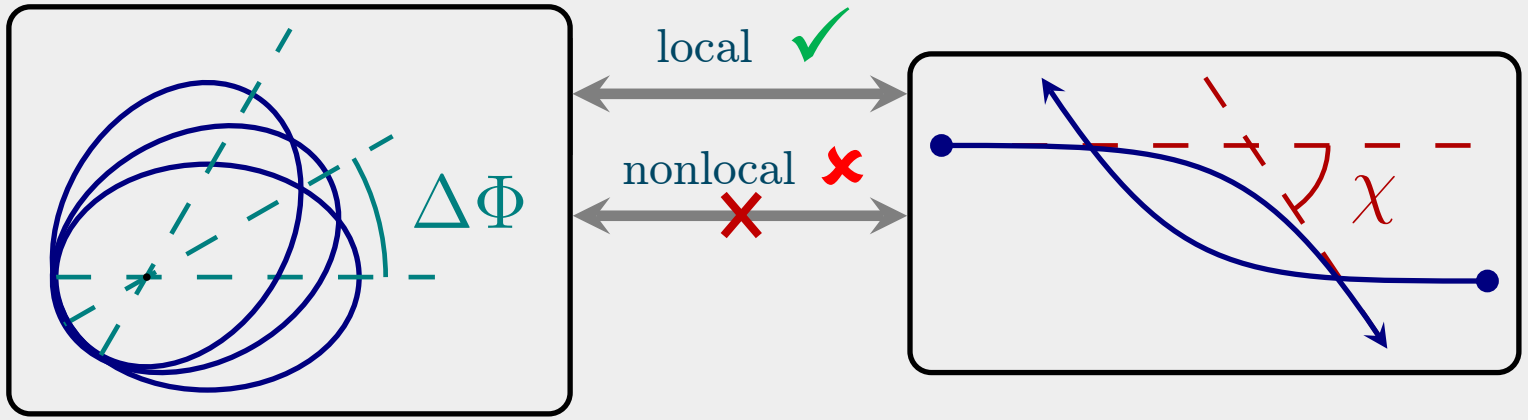


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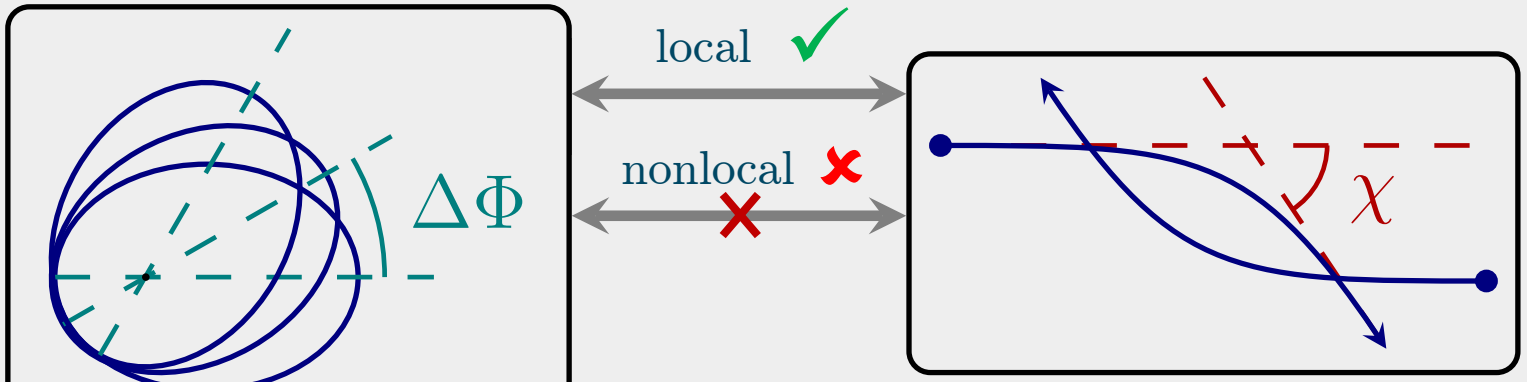


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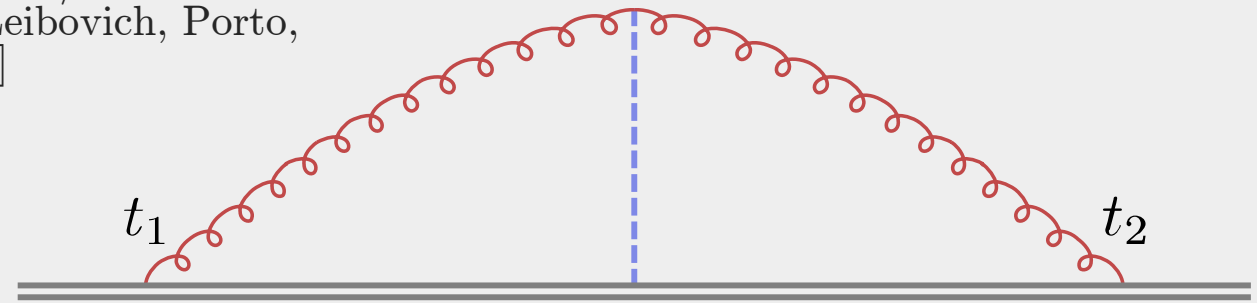
[Cho, Kälin, Porto, '22 / Damour, Jaranowski, Schäfer, '14 / Galley, Leibovich, Porto, Ross, '15]



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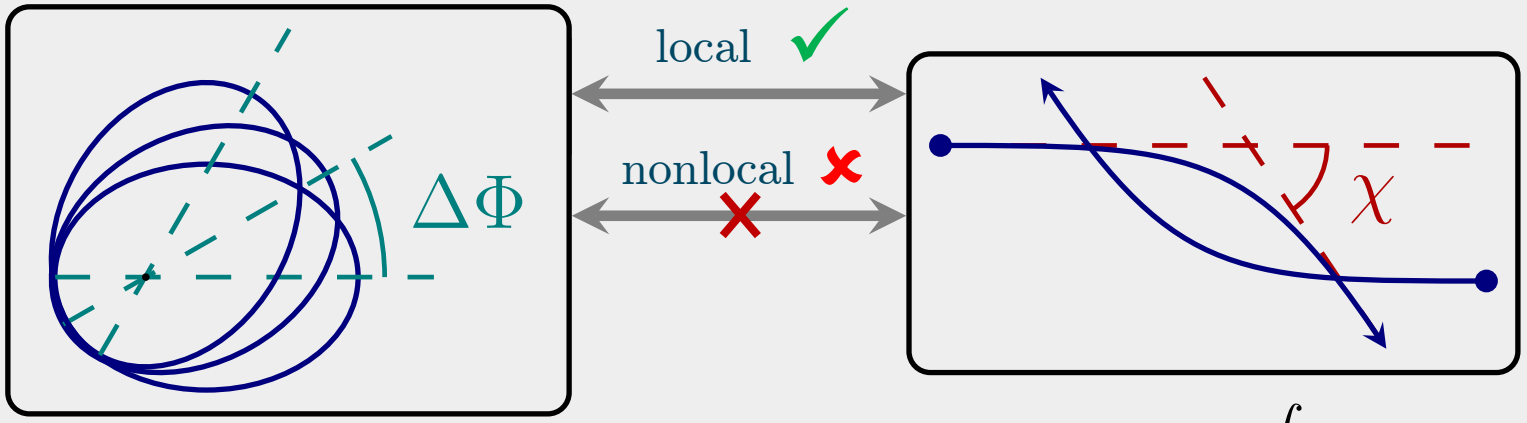


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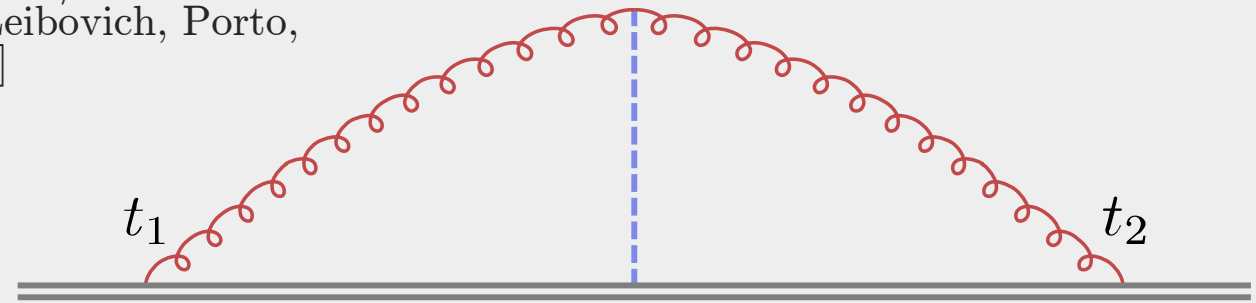


$$\mathcal{S}_r^{\rm nloc} \sim GE \int \mathcal{F}(t_1,t_2) \frac{{\rm d}t_1 {\rm d}t_2}{|t_1-t_2|},$$

$$\sim GE \int \frac{{\rm d}E}{{\rm d}\omega} \log(\omega^2)\, {\rm d}\omega$$

$$\chi \sim \partial_j \mathcal{S}_r$$

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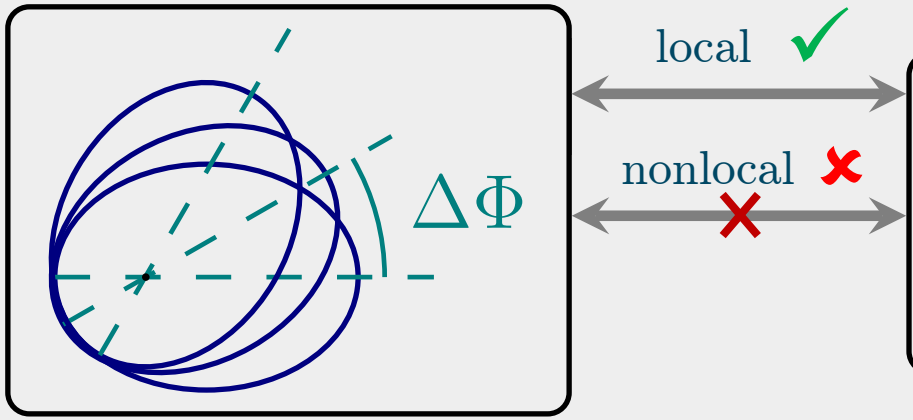
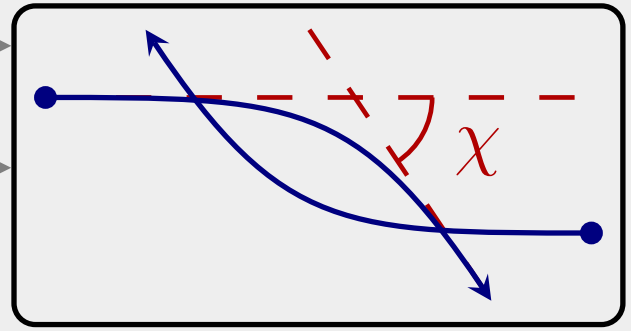



local ✓

nonlocal ✗




$\Delta p_1 \to \Delta E$

$\to \chi$

[Cho, Kälin, Porto, '22 / Damour, Jaranowski, Schäfer, '14 / Galley, Leibovich, Porto, Ross, '15]

$$\mathcal{S}_r^{\rm nloc} \sim GE \int \mathcal{F}(t_1,t_2)\frac{{\rm d}t_1{\rm d}t_2}{|t_1-t_2|},$$

$$\sim GE \int \frac{{\rm d}E}{{\rm d}\omega}\log(\omega^2)\,{\rm d}\omega$$

$$\chi \sim \partial_j \mathcal{S}_r \qquad u_{\rm com} \equiv \frac{m_1u_1+m_2u_2}{E},$$

- Related to previous computation:

$$\Delta E = \int \frac{{\rm d}E}{{\rm d}\omega}{\rm d}\omega,$$

radiation mode: $k=(\omega,\boldsymbol{k}),$

$\omega = k\cdot u_{\rm com}$

$\Rightarrow$ add to integrand: $\log[(k\cdot u_{\rm com})^2]$

# Nonlocal-in-time integrals at 4PM

- Nonlocal action: $\mathcal{S}_r^{\text{nonloc}} \sim GE \int \frac{\mathrm{d}E}{\mathrm{d}\omega} \log(\omega^2)\, \mathrm{d}\omega,$ $\omega = k \cdot u_{\text{com}}$

- Example integral family:

$$D = 4 - 2\epsilon$$

$$I_{\log}(\gamma, q) = \int \mathrm{d}^D\ell_1 \mathrm{d}^D\ell_2 \frac{\delta(\ell_1 \cdot u_1)\delta(\ell_2 \cdot u_2)}{(\ell_1 \cdot u_2)(\ell_2 \cdot u_1)} \frac{1}{\ell_1^2 \ell_2^2 k^2} \log[(k \cdot u_{\text{com}})^2] \qquad u_{\text{com}} \equiv \frac{m_1 u_1 + m_2 u_2}{E},$$

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$$= I(\gamma) + \tilde{\epsilon}\, I_{\log}(\gamma, q) + \mathrm{O}(\tilde{\epsilon}^2)$$

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$$D = 4 - 2\epsilon$$

$$u_{\rm com} \equiv \frac{m_1 u_1 + m_2 u_2}{E},$$

$$= \frac{m_1}{E}(u_1 + q u_2),$$

$$u_1 \cdot u_2 = \gamma, \qquad q = \frac{m_2}{m_1}$$

# IBP reduction

- Integration-by-parts reduction to basis

$$\vec{f} = \begin{pmatrix} I_{11-10001110} \\ \vdots \\ I_{1110001110} \end{pmatrix}, \qquad I_{a_1 \dots a_{10}} = \sum_{i=1}^{17} c_i(x, q, \epsilon) f_i$$

  - LiteRed to generate IBP without $\tilde{\epsilon}$ $\qquad [(k \cdot u_{\text{com}})^2]^{\tilde{\epsilon}}$
  - manually shift $\quad a_{10} \to a_{10} - 2\tilde{\epsilon}$
  - solve with FiniteFlow

# Differential equations and boundary conditions

- Two variables: step-by-step approach

$$\partial_q \vec{f} = A_q(x, q, \epsilon)\vec{f},$$

$$\partial_x \vec{f} = A_x(x, q, \epsilon)\vec{f}$$

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- Two variables: step-by-step approach

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$$[(k \cdot u_{\text{com}})^2]^{\tilde{\epsilon}} = [(\ell_2 \cdot u_1 + q\,\ell_1 \cdot u_2)^2]^{\tilde{\epsilon}}$$

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$$[(k \cdot u_{\text{com}})^2]^{\tilde{\epsilon}} = [(\ell_2 \cdot u_1 + q\,\ell_1 \cdot u_2)^2]^{\tilde{\epsilon}} = [(\ell_2 \cdot u_1)^2]^{\tilde{\epsilon}} + 2q\,\tilde{\epsilon}\frac{(\ell_1 \cdot u_2)(\ell_2 \cdot u_1)}{[(\ell_2 \cdot u_1)^2]^{1-\tilde{\epsilon}}} + \mathcal{O}(q^2)$$

$$= \sum_{n=0}^{\infty} q^n \frac{(-1)^n}{n!} \frac{\Gamma(n - 2\tilde{\epsilon})}{\Gamma(-2\tilde{\epsilon})} \frac{[(\ell_1 \cdot u_2)(\ell_2 \cdot u_1)]^n}{[(\ell_2 \cdot u_1)^2]^{n-\tilde{\epsilon}}}$$

"old" integrals but
with shifted power

# Regular basis

- Two variables: step-by-step approach

$$\partial_q \vec{f} = A_q(x,q,\epsilon)\vec{f},$$

$$\partial_x \vec{f} = A_x(x,q,\epsilon)\vec{f}$$

main interest: “resum” q expansion

  - boundary constants from q expansion with full x dependence

- Solve through $\epsilon, \tilde{\epsilon}$ regular basis $\vec{g}$ [Lee, Onishchenko, ’19 / Tancredi, ‘15]

1) $$\vec{g} = \vec{g}^{(0,0)} + \tilde{\epsilon}\,\vec{g}^{(0,1)} + \mathcal{O}(\epsilon, \tilde{\epsilon}^2)$$

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regular integrand

$$= \sum_{i=1}^{17} [c_i^{(0)} g_i^{(0)} + \tilde{\epsilon}(c_i^{(1)} g_i^{(0)} + c_i^{(0)} g_i^{(1)})]$$

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$$= \Delta E + \tilde{\epsilon}\,\mathcal{S}_r^{\text{nloc}}$$

# Differential equations

- Solve through $\epsilon, \tilde{\epsilon}$ regular basis $\vec{g}$ [Lee, Onishchenko, '19 / Tancredi, '15]

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- How to find this basis? $\vec{g} = \vec{g}^{(0)} + \tilde{\epsilon}\, \vec{g}^{(1)}$
  - idea: all $g_i^{(0)}$ are linearly independent, no cancellations possible
    $\Rightarrow$ eliminate all linear relations
  - find through integrand and differential equations

$$\mathcal{I}(\epsilon, \tilde{\epsilon}) = \frac{1}{\tilde{\epsilon}} \sum_{i=1}^{17} c_i^{(-1)} f_i + \mathcal{O}(\tilde{\epsilon}^0) \longrightarrow \sum_{i=1}^{17} c_i^{(-1)} f_i^{(0)} = 0 \longrightarrow g_j = \frac{1}{\tilde{\epsilon}} \sum_{i=1}^{17} c_i^{(-1)} f_i = \mathcal{O}(\tilde{\epsilon}^0)$$

# Leading order integration: polylogarithms

$$\partial_q \vec{g} = (\tilde{A}_q^{(0)} + \tilde{\epsilon}\,\tilde{A}_q^{(1)})\vec{g}, \qquad \vec{g} = \vec{g}^{(0)} + \tilde{\epsilon}\,\vec{g}^{(1)}$$

- diagonal blocks first
  - ➢ derive and solve Picard-Fuchs operator:

$$\tilde{A}_q^{(0)} = \begin{pmatrix} \boxed{D_1} & 0 & \dots & 0 \\ C_{2,1} & \boxed{D_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \dots & \boxed{D_n} \end{pmatrix}$$

$$\mathcal{L}_q^n g_j^{(0)} = 0, \qquad \mathcal{L}_q^n = \sum_{m=0}^{n} b_m(x,q)\frac{\partial^m}{\partial q^m}$$

[Adams, Chaubey, Weinzierl, '14]

maple

polylogarithmic: $= (\partial_q + R_1)(\partial_q + R_2)\dots(\partial_q + R_n)$

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rationalize square-root $\frac{E}{m_1} = \sqrt{(q+x)(q+\frac{1}{x})}$ through $q^{-1} = -\gamma - v\frac{1}{2}\left(y + \frac{1}{y}\right)$

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- off-diagonal blocks

$$G^{(0)} = \mathrm{P}\, e^{\int \tilde{A}_y^{(0)} \mathrm{d}y}, \qquad \vec{g}^{(0)} = G^{(0)}\vec{b}^{(0)}$$

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$$= \mathbb{1} + \int_0^y \mathrm{d}y_1 \tilde{A}_{y_1}^{(0)} + \int_0^y \mathrm{d}y_2 \tilde{A}_{y_2}^{(0)} \int_0^{y_2} \mathrm{d}y_1 \tilde{A}_{y_1}^{(0)} + \dots$$

iterated integration in y with PolyLogTools (double poles, products)

letters: $\{y, 1+y, 1-y, y - \frac{1+x}{1-x}, y - \frac{1-x}{1+x}, 1 + 2\frac{1-x}{1+x}y + y^2\}$

# Leading order integration: elliptics

$$\partial_q \vec{g} = (\tilde{A}_q^{(0)} + \tilde{\epsilon}\,\tilde{A}_q^{(1)})\vec{g}, \qquad \mathcal{L}_q^n g_j^{(0)} = 0,$$

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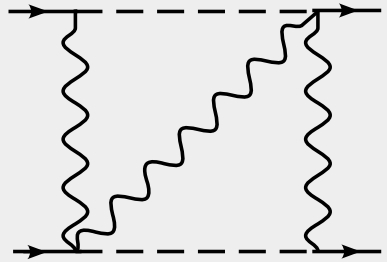

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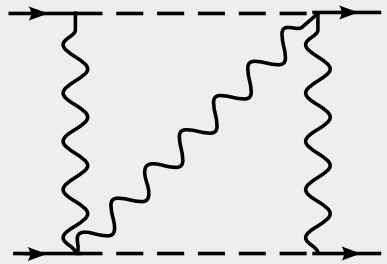

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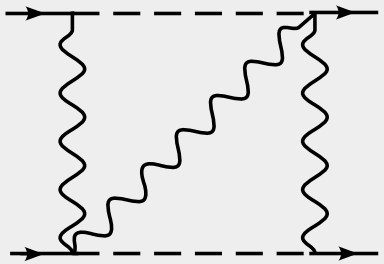

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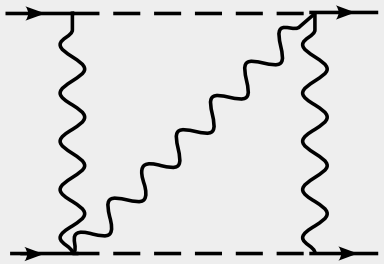

- Analyzing the 4x4 DE:
  - expansion: $\vec{g}^{(0)} \simeq P(q,x) q^{A_0^{(0)}} \vec{b}(x), \qquad q^{A_0^{(0)}} = \begin{pmatrix} 1 & \log q & \frac{\log^2 q}{2} & 0 \\ 0 & 1 & \log q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$P(q,x) = 1 + \sum_{i=1}^{\infty} q^i P^{(i)}(x)$

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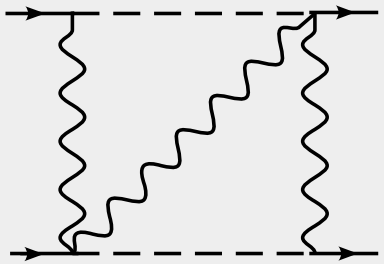

- Analyzing the 4x4 DE:
  - expansion: $\vec{g}^{(0)} \simeq P(q,x) q^{A_0^{(0)}} \vec{b}(x),$

$$P(q,x) = 1 + \sum_{i=1}^{\infty} q^i P^{(i)}(x)$$

$$q^{A_0^{(0)}} = \begin{pmatrix} 1 & \log q & \frac{\log^2 q}{2} & 0 \\ 0 & 1 & \log q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

# Leading order integration: elliptics

$$\partial_q \vec{g} = (\tilde{A}_q^{(0)} + \tilde{\epsilon}\,\tilde{A}_q^{(1)})\vec{g}, \qquad \mathcal{L}_q^n g_j^{(0)} = 0,$$

- $D_1$ not first-order factorizable:

$$\vec{g}_{D_1} = \begin{pmatrix} I_{00001111} \\ I_{00-101111} \\ I_{000-11111} \\ I_{00002111} \\ \tilde{\epsilon}^{-1} I_{-10002110} \\ \tilde{\epsilon}^{-1} I_{0-1002110} \end{pmatrix},$$

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$$D_1 = \begin{pmatrix} \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \end{pmatrix}$$

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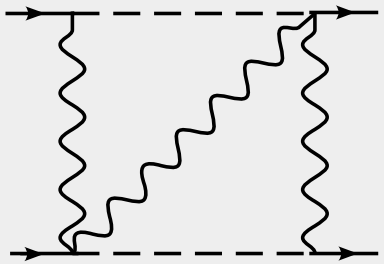

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$P(q,x) = 1 + \sum_{i=1}^{\infty} q^i P^{(i)}(x)$

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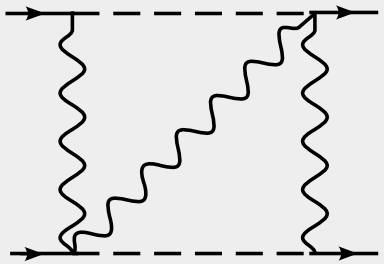

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$$x^{A_0^{(0)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \log x & \frac{\log^2 x}{2} \\ 0 & 0 & 1 & \log x \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  - Baikov rep: $g_1^{(0)} \sim \int \frac{\mathrm{d}w_1 \mathrm{d}w_2}{\sqrt{P_{4,3}(w_1,w_2)}} = \int \frac{\mathrm{d}w_1 \mathrm{d}w_2}{\sqrt{4 + v_\infty^2 w_1^2}\sqrt{w_1^2 + 4w_2}\sqrt{4\gamma q w_2 + v_\infty^2 w_2^2 + 4q^2(1+w_2)}}$

# Leading order integration: elliptics

- Analyzing the 4x4 DE:
  - change of variables: $x = \alpha_1 t, \quad q = \alpha_2 t$ $\longrightarrow$ $\partial_t \vec{g} = \tilde{A}_t^{(0)} \vec{g},$

$$\tilde{A}_t^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

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  - choose $\beta_1 = qx, \quad \partial_{\beta_1} \vec{g} = \tilde{A}_{\beta_1}^{(0)} \vec{g},$
    $\beta_2 = \frac{q}{x}, \quad \partial_{\beta_2} \vec{g} = \tilde{A}_{\beta_2}^{(0)} \vec{g}$

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$$\tilde{A}_{\beta_1}^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

$$\tilde{A}_{\beta_2}^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

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$$\tilde{A}_{\beta_1}^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

two identical elliptic sectors

$$\tilde{A}_{\beta_2}^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

$\mathrm{K}(-qx)$ $\quad$ $\mathrm{K}(-qx)$

$\mathrm{K}(1+qx)$ $\quad$ $\mathrm{K}(1+qx)$

# Leading order integration: elliptics

- Analyzing the 4x4 DE:
  - change of variables: $x = \alpha_1 t, \quad q = \alpha_2 t$ $\longrightarrow$ $\partial_t \vec{g} = \tilde{A}_t^{(0)} \vec{g},$

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$$\tilde{A}_{\beta_1}^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

two identical elliptic sectors

$\mathrm{K}(-qx)$, $\mathrm{K}(1+qx)$ | $\mathrm{K}(-qx)$, $\mathrm{K}(1+qx)$

$$\tilde{A}_{\beta_2}^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

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# Leading order integration: elliptics

- Analyzing the 4x4 DE:
  - change of variables: $x = \alpha_1 t, \quad q = \alpha_2 t$ $\longrightarrow$ $\partial_t \vec{g} = \tilde{A}_t^{(0)} \vec{g},$ $\tilde{A}_t^{(0)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$
  - choose $\beta_1 = qx, \quad \partial_{\beta_1} \vec{g} = \tilde{A}_{\beta_1}^{(0)} \vec{g},$ $\beta_2 = \frac{q}{x}, \quad \partial_{\beta_2} \vec{g} = \tilde{A}_{\beta_2}^{(0)} \vec{g}$

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two identical elliptic sectors

$\mathrm{K}(-\frac{q}{x})$ $\quad$ $\mathrm{K}(-\frac{q}{x})$

$\mathrm{K}(1+\frac{q}{x})$ $\quad$ $\mathrm{K}(1+\frac{q}{x})$

  - four independent solutions: $\mathrm{K}(-qx)\mathrm{K}(-\frac{q}{x}), \mathrm{K}(-qx)\mathrm{K}(1+\frac{q}{x}),$ $\mathrm{K}(1+qx)\mathrm{K}(-\frac{q}{x}), \mathrm{K}(1+qx)\mathrm{K}(1+\frac{q}{x})$

# Integration continued

$$\partial_q \vec{g} = (\tilde{A}_q^{(0)} + \tilde{\epsilon}\,\tilde{A}_q^{(1)})\vec{g}, \qquad \vec{g} = \vec{g}^{(0)} + \tilde{\epsilon}\,\vec{g}^{(1)}$$

- Leading order
  - diagonal blocks first
    - ➢ derive and solve Picard-Fuchs operator:
  - off-diagonal blocks

$$\tilde{A}_q^{(0)} = \begin{pmatrix} \boxed{D_1} & 0 & \dots & 0 \\ C_{2,1} & \boxed{D_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \dots & \boxed{D_n} \end{pmatrix}$$

$$G^{(0)} = \mathrm{P}\, e^{\int \tilde{A}_q^{(0)} \mathrm{d}q}, \qquad \vec{g}^{(0)} = G^{(0)} \vec{b}^{(0)}$$

$$= \mathbb{1} + \int_0^q \mathrm{d}q_1 \tilde{A}_{q_1}^{(0)} + \int_0^q \mathrm{d}q_2 \tilde{A}_{q_2}^{(0)} \int_0^{q_2} \mathrm{d}q_1 \tilde{A}_{q_1}^{(0)} + \dots$$

- iterated integration in y with PolyLogTools

- iterated integrals in q for elliptic kernels

# Functions in the result

- Polylogs: $\operatorname{arctanh}(x), \log(x), \log\left(\frac{1+x}{2}\right), \operatorname{Li}_2(x), \operatorname{Li}_2(\frac{x-1}{1+x}), \operatorname{Li}_2(\frac{1+x}{2}), \operatorname{Li}_2(1-x^2),$

  $\log(q), \log(y), \log(1-y), \log(1+y), \log(1+x+y-xy), \log(1-x+y+xy), \log(1+x+2y-2xy+y^2+xy^2)$

  $\operatorname{Li}_2(y), \operatorname{Li}_2(-y), \operatorname{Li}_2(\frac{(1-x)(1-y)}{2}), \operatorname{Li}_2(\frac{(1+x)(1-y)}{2}), \operatorname{Li}_2(\frac{(-1+x)(1+y)}{2x}),$

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- Elliptic II-kernels: $f_1 = \frac{\mathrm{K}(-qx)\mathrm{K}(1+\frac{q}{x})-\mathrm{K}(-\frac{q}{x})\mathrm{K}(1+qx)}{\pi},$

$$f_2 = \frac{f_1}{q}, \quad f_3 = \partial_x f_1, \quad f_4 = \frac{\partial_x f_1}{q}, \quad f_5 = \left[\frac{1-x^2}{x}(1+q\partial_q) - \frac{1-q^2}{q}x\partial_x\right] \frac{f_1}{\sqrt{(q+x)(q+\frac{1}{x})}}$$

# Functions in the result

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$$E \sim \sqrt{(q+x)(q+\tfrac{1}{x})}$$

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note: $\left[\frac{1-x^2}{x}q\partial_q - \frac{1-q^2}{q}x\partial_x\right][(q+x)(q+\frac{1}{x})]^a = a\frac{1-x^2}{x}[(q+x)(q+\frac{1}{x})]^a \sim E\partial_E + \ldots$

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note: $\left[\frac{1-x^2}{x}q\partial_q - \frac{1-q^2}{q}x\partial_x\right][(q+x)(q+\frac{1}{x})]^a = a\frac{1-x^2}{x}[(q+x)(q+\frac{1}{x})]^a \sim E\partial_E + \ldots$

- IIs: $\mathrm{II}(f_1), \quad \mathrm{II}(f_2), \quad \mathrm{II}(f_3), \quad \mathrm{II}(f_4), \quad \mathrm{II}(f_5)$ transcendental weight unclear

$$\mathrm{II}(\tfrac{1}{q}, f_3), \quad \mathrm{II}(\tfrac{1}{q}, f_4), \quad \mathrm{II}\Big(\frac{1}{\sqrt{(q+x)(q+\frac{1}{x})}}, f_5\Big), \quad \mathrm{II}\Big(\frac{1}{q\sqrt{(q+x)(q+\frac{1}{x})}}, f_5\Big)$$

# Post-Newtonian expansion

- Elliptics:

$$f_1 = \frac{\mathrm{K}(-qx)\mathrm{K}(1+\frac{q}{x})-\mathrm{K}(-\frac{q}{x})\mathrm{K}(1+qx)}{\pi},$$

$$= (1-x)\frac{\mathrm{K}(-q)\mathrm{K}(1+q)-\mathrm{K}(-q)\mathrm{E}(1+q)-\mathrm{E}(-q)\mathrm{K}(1+q)}{\pi(1+q)} + \mathcal{O}((1-x)^2),$$

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legendre relation

$$= (1-x)\frac{-1}{2(1+q)} + \mathcal{O}((1-x)^2)$$

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legendre relation

$$= (1-x)\frac{-1}{2(1+q)} + \mathcal{O}((1-x)^2)$$

- Comparison with 6PN literature:

[Bini, Damour, Geralico, '20]

$$W_1^{\text{tail}} = \frac{2}{15}M^2\nu^2\pi\Bigg\{v^4\left[100+37\log\left(\frac{sv^2}{4j}\right)\right]$$
$$+v^6\left[\frac{137}{4}-\frac{1338}{7}\nu+\left(\frac{1357}{56}-\frac{111}{2}\nu\right)\log\left(\frac{sv^2}{4j}\right)\right]$$
$$+v^8\left[\frac{826619}{8064}-\frac{4727}{144}\nu+\frac{1080733}{4032}\nu^2+\left(\frac{27953}{672}-\frac{2517}{112}\nu+\frac{555}{8}\nu^2\right)\log\left(\frac{sv^2}{4j}\right)\right]$$
$$+\mathcal{O}(v^{10})\Bigg\}$$

$$M = m_1 + m_2,$$
$$\nu = \frac{m_1 m_2}{M^2},$$
$$= \frac{q}{(1+q)^2}$$

# Summary

- Non-local integration is a two-scale problem
- Elliptics appear in intriguing way already at two-loops

$$\beta_1 = qx, \qquad \beta_2 = \frac{q}{x},$$

$$\tilde{A}^{(0)}_{\beta_1} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}, \qquad \tilde{A}^{(0)}_{\beta_2} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

- Regular-basis simplifies the computation for manifestly finite quantities/observables

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# Questions

- Transcendental weight not satisfied?
- Canonical form might be necessary

$$\beta_1 = qx,$$
$$\beta_2 = \frac{q}{x},$$

- Best choice of variables?
  - very simple PN structure
  - reduce to single-variable problem?
- Is the total bound local+non-local also a two-variable problem?