Classifying the geometries of post-Minkowskian integrals

Roger Morales

Niels Bohr Institute

August 7, 2024 @ Elliptics and Beyond '24 (MIAPbP)

[2312.11371], [2405.17255] with H. Frellesvig, and M. Wilhelm

[24xx.xxxxx] with H. Frellesvig, S. Pögel, S. Weinzierl and M. Wilhelm

[24yy.yyyyy] with D. Brammer, H. Frellesvig and M. Wilhelm

Motivation

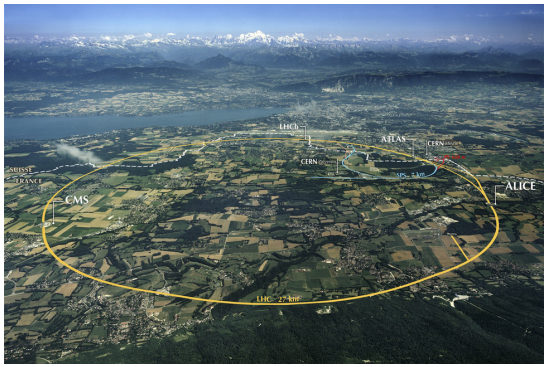


Image: CERN

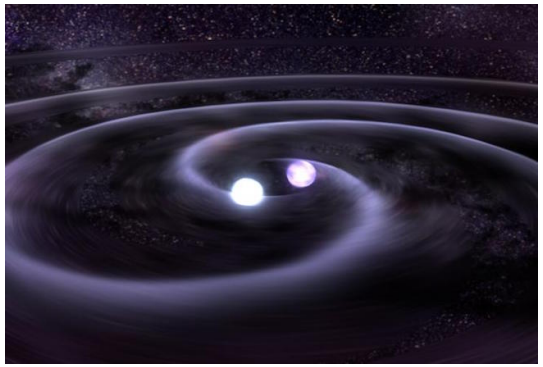


Image: NASA/Goddard Space Flight Center

New experiments $\Longrightarrow$ Need for precision predictions!

Motivation

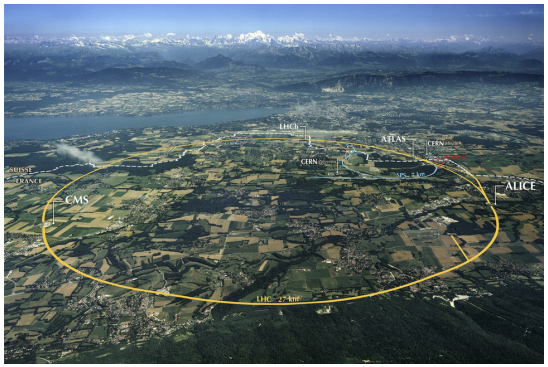


Image: CERN

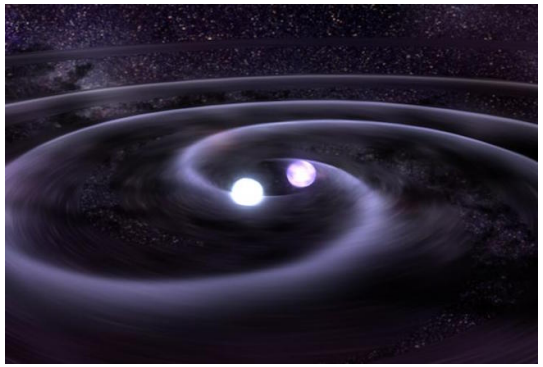
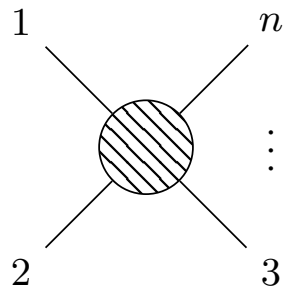


Image: NASA/Goddard Space Flight Center

New experiments $\Longrightarrow$ Need for precision predictions!

Scattering amplitude $\mathcal{A}$



Motivation

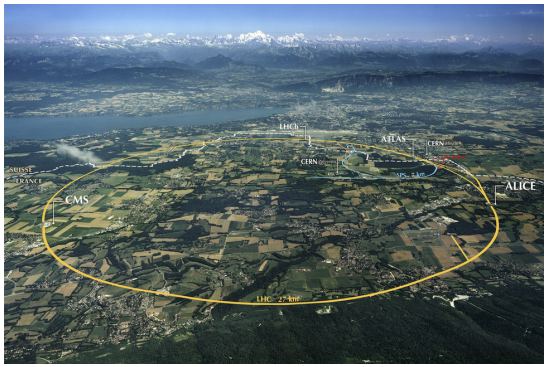


Image: CERN

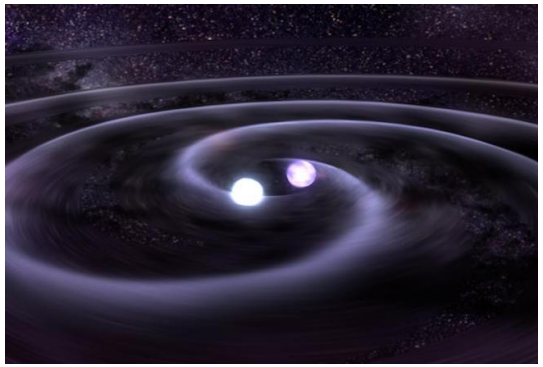
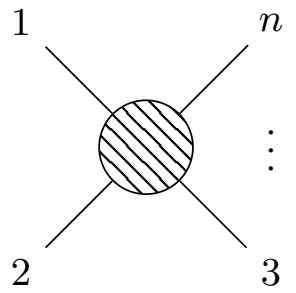


Image: NASA/Goddard Space Flight Center

New experiments $\Longrightarrow$ Need for precision predictions!

Scattering amplitude $\mathcal{A}$



Colliders:

$$\sigma \sim \int d(\text{phase space})\, |\mathcal{A}|^2$$

Motivation

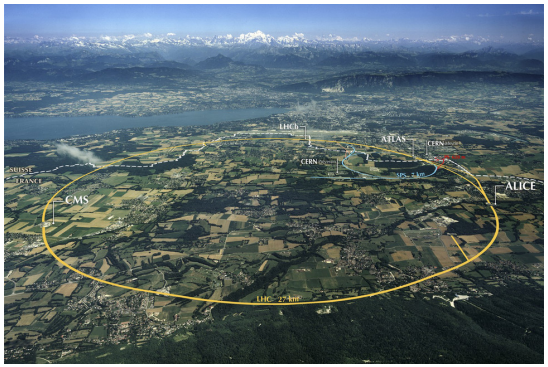


Image: CERN

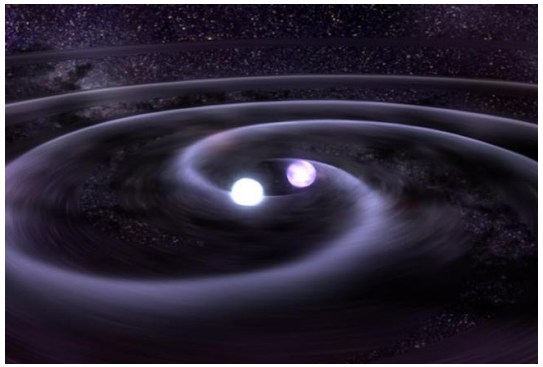
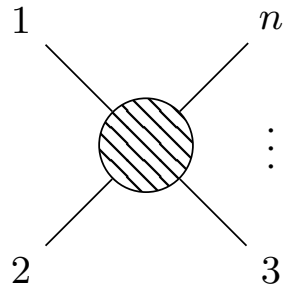


Image: NASA/Goddard Space Flight Center

New experiments $\Longrightarrow$ Need for precision predictions!

Scattering amplitude $\mathcal{A}$



Colliders:

$$\sigma \sim \int d(\text{phase space})\,|\mathcal{A}|^2$$

EOB [Buonanno, Damour (1999,2000)]

Gravitational waves:

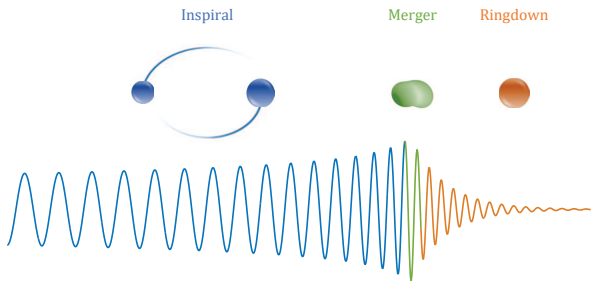


Image: [1610.03567]

Outline

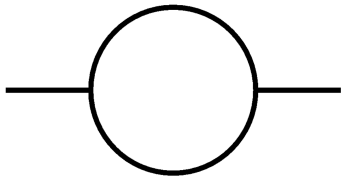
1. Geometries in Feynman integrals
2. How to detect non-trivial geometries
3. Results: Classification of PM geometries
4. Conclusions and outlook

Outline

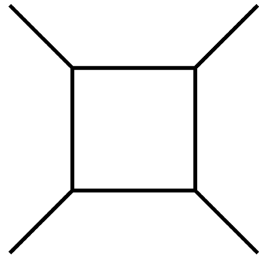
1. Geometries in Feynman integrals
2. How to detect non-trivial geometries
3. Results: Classification of PM geometries
4. Conclusions and outlook

One loop and multiple polylogarithms

At one loop



$$\overset{D=2}{\sim} \log(\dots),$$



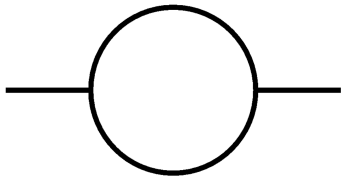
$$\overset{D=4}{\sim} \mathrm{Li}_2(\dots) + \dots$$

Classical polylogarithms

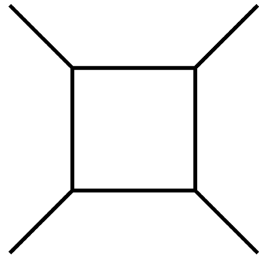
$$\mathrm{Li}_n(x) \equiv \int_0^x \frac{dt}{t}\,\mathrm{Li}_{n-1}(t) \quad \text{with} \quad \mathrm{Li}_1(x) \equiv -\log(1-x) = \int_0^x \frac{dt}{1-t}$$

One loop and multiple polylogarithms

At one loop



$\overset{D=2}{\sim} \log(\dots)$,



$\overset{D=4}{\sim} \mathrm{Li}_2(\dots) + \dots$

Classical polylogarithms

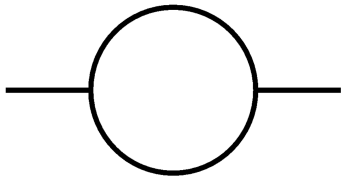
$$\mathrm{Li}_n(x) \equiv \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \quad \text{with} \quad \mathrm{Li}_1(x) \equiv -\log(1-x) = \int_0^x \frac{dt}{1-t}$$

$\hookrightarrow$ In general: All 1-loop Feynman integrals are described by multiple polylogarithms (MPLs) [Chen (1977)], [Goncharov (1995)], ...

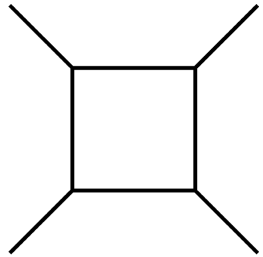
$$G(a_1, \dots, a_n; x) \equiv \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n; t), \qquad G(; x) = 1$$

One loop and multiple polylogarithms

At one loop



$\overset{D=2}{\sim} \log(\dots)$,



$\overset{D=4}{\sim} \mathrm{Li}_2(\dots) + \dots$

Classical polylogarithms

$$\mathrm{Li}_n(x) \equiv \int_0^x \frac{dt}{t}\,\mathrm{Li}_{n-1}(t) \quad \text{with} \quad \mathrm{Li}_1(x) \equiv -\log(1-x) = \int_0^x \frac{dt}{1-t}$$

$\hookrightarrow$ In general: All 1-loop Feynman integrals are described by multiple polylogarithms (MPLs) [Chen (1977)], [Goncharov (1995)], ...

$$G(a_1,\dots,a_n;x) \equiv \int_0^x \frac{dt}{t-a_1} G(a_2,\dots,a_n;t), \qquad G(;x) = 1$$

$\Longrightarrow$ But scattering amplitudes are richer than MPLs!

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Consider a Feynman (parameter) integral

$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)}$$

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Consider a Feynman (parameter) integral

$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)} = \int_0^1 dx_2 \left(\frac{\log(1 - 1/\sqrt{P(x_2)})}{2\sqrt{P(x_2)}} - (- \to +) \right)$$

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Consider a Feynman (parameter) integral

$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)} = \int_0^1 dx_2 \left(\frac{\log(1 - 1/\sqrt{P(x_2)})}{2\sqrt{P(x_2)}} - (- \to +) \right)$$

What is the geometry of $y^2 = P(x_2)$?

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Consider a Feynman (parameter) integral

$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)} = \int_0^1 dx_2 \left(\frac{\log(1 - 1/\sqrt{P(x_2)})}{2\sqrt{P(x_2)}} - (- \to +) \right)$$

What is the geometry of $y^2 = P(x_2)$?

- $\deg P(x_2) \leq 2 \implies$ Change of variables $\sqrt{P(x_2)} \to \sqrt{(\dots)^2}$

  $\hookrightarrow$ MPL $\implies$ Integral over the Riemann sphere

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Consider a Feynman (parameter) integral

$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)} = \int_0^1 dx_2 \left(\frac{\log(1 - 1/\sqrt{P(x_2)})}{2\sqrt{P(x_2)}} - (- \to +) \right)$$

What is the geometry of $y^2 = P(x_2)$?

- $\deg P(x_2) \geq 3 \implies \sqrt{P(x_2)}$ cannot be rationalized

  $\hookrightarrow$ Integral over a non-trivial geometry!

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

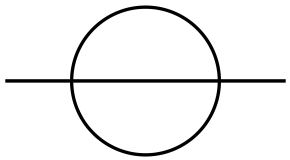
Consider a Feynman (parameter) integral

$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)} = \int_0^1 dx_2 \left(\frac{\log(1 - 1/\sqrt{P(x_2)})}{2\sqrt{P(x_2)}} - (- \to +) \right)$$

What is the geometry of $y^2 = P(x_2)$?

- $\deg P(x_2) \geq 3 \implies \sqrt{P(x_2)}$ cannot be rationalized

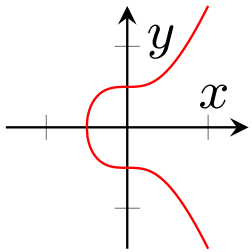
  $\hookrightarrow$ Integral over a non-trivial geometry!



[Broadhurst, Fleischer, Tarasov (1993)], ...,
[Adams, Bogner, Weinzierl (2013)], ...

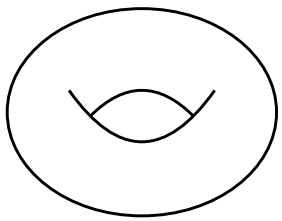
$\deg P(x_2) = 3, 4 \implies$

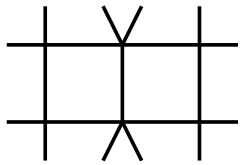
Elliptic curve



$=$

Torus





[Caron-Huot, Larsen (2012)], ...,
[**RM**, Spiering, Wilhelm, Yang, Zhang (2022)]

[see talks by Anne, Christoph, Ekta, Yoann, ...]

Beyond multiple polylogarithms

How do integrals beyond multiple polylogarithms occur?

Consider a Feynman (parameter) integral

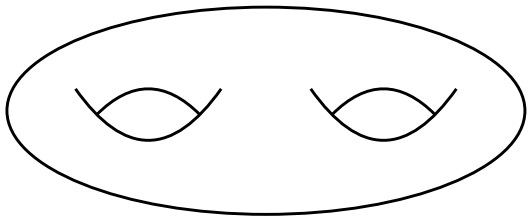
$$I = \int_0^1 dx_1 dx_2 \, \frac{1}{{x_1}^2 - P(x_2)} = \int_0^1 dx_2 \left(\frac{\log(1 - 1/\sqrt{P(x_2)})}{2\sqrt{P(x_2)}} - (- \to +) \right)$$

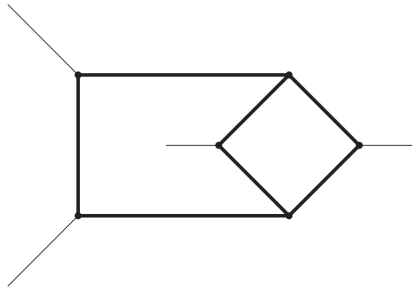
What is the geometry of $y^2 = P(x_2)$?

- $\deg P(x_2) \geq 3 \implies \sqrt{P(x_2)}$ cannot be rationalized

  $\hookrightarrow$ Integral over a non-trivial geometry!

$\deg P(x_2) = 5, 6 \implies$ 2-torus





[see talks by Federico, Franziska, Sebastian, . . .]

[Georgoudis, Zhang (2015)],
[Marzucca, McLeod, Page, Pögel, Weinzierl (2023)]

Calabi-Yau manifolds

In general we have more than 2 variables:

$$I = \int_0^1 \frac{dx_1 \cdots dx_n}{{x_1}^2 - P(x_2, \ldots, x_n)} = \int_0^1 dx_2 \cdots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (- \to +) \right)$$

Calabi-Yau manifolds

In general we have more than 2 variables:

$$I = \int_0^1 \frac{dx_1 \cdots dx_n}{{x_1}^2 - P(x_2, \ldots, x_n)} = \int_0^1 dx_2 \cdots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (- \to +) \right)$$

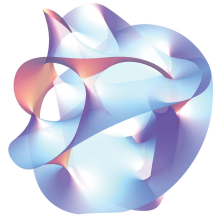
- Geometry of $y^2 = P(x_2, \ldots, x_n) \Longrightarrow n - 1$ dim. manifold

Calabi-Yau manifolds

In general we have more than 2 variables:

$$I = \int_0^1 \frac{dx_1 \cdots dx_n}{{x_1}^2 - P(x_2, \ldots, x_n)} = \int_0^1 dx_2 \cdots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (- \to +) \right)$$

- Geometry of $y^2 = P(x_2, \ldots, x_n) \Longrightarrow n - 1$ dim. manifold
- If $\deg P(x_2, \ldots, x_n) = 2n \Longrightarrow$ Calabi-Yau $(n-1)$-fold

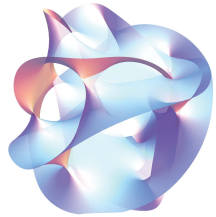


Calabi-Yau manifolds

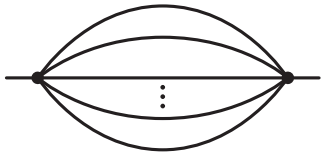
In general we have more than 2 variables:

$$I = \int_0^1 \frac{dx_1 \cdots dx_n}{{x_1}^2 - P(x_2, \ldots, x_n)} = \int_0^1 dx_2 \cdots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (- \to +) \right)$$

- Geometry of $y^2 = P(x_2, \ldots, x_n) \Longrightarrow n - 1$ dim. manifold
- If $\deg P(x_2, \ldots, x_n) = 2n \Longrightarrow$ Calabi-Yau $(n-1)$-fold

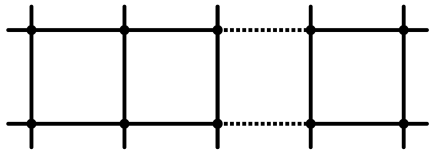


Examples:



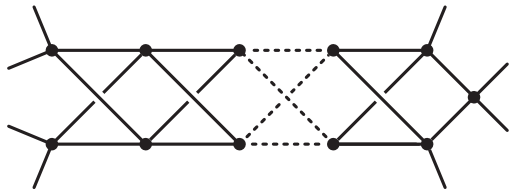
CY $(L-1)$-fold

[Broadhurst, Fleischer, Tarasov (1993)], ..., [Bönisch, Duhr, Fischbach, Klemm, Nega (2021)]



CY $(L-1)$-fold

[Bourjaily, He, McLeod, von Hippel, Wilhelm (2018)], ..., [Cao, He, Tang (2023)]



CY $(2L-2)$-fold

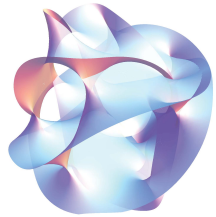
[Bourjaily, McLeod, von Hippel, Wilhelm (2018)], [Lairez, Vanhove (2022)]

Calabi-Yau manifolds

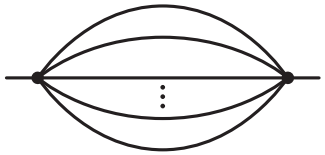
In general we have more than 2 variables:

$$I = \int_0^1 \frac{dx_1 \cdots dx_n}{{x_1}^2 - P(x_2, \ldots, x_n)} = \int_0^1 dx_2 \cdots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (- \to +) \right)$$

- Geometry of $y^2 = P(x_2, \ldots, x_n) \Longrightarrow n-1$ dim. manifold
- If $\deg P(x_2, \ldots, x_n) = 2n \Longrightarrow$ Calabi-Yau $(n-1)$-fold

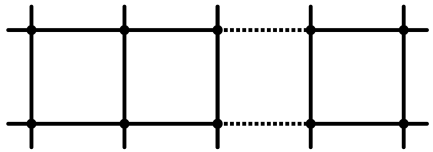


Examples:



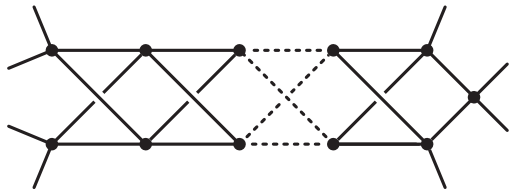
CY $(L-1)$-fold

[Broadhurst, Fleischer, Tarasov (1993)], ..., [Bönisch, Duhr, Fischbach, Klemm, Nega (2021)]



CY $(L-1)$-fold

[Bourjaily, He, McLeod, von Hippel, Wilhelm (2018)], ..., [Cao, He, Tang (2023)]



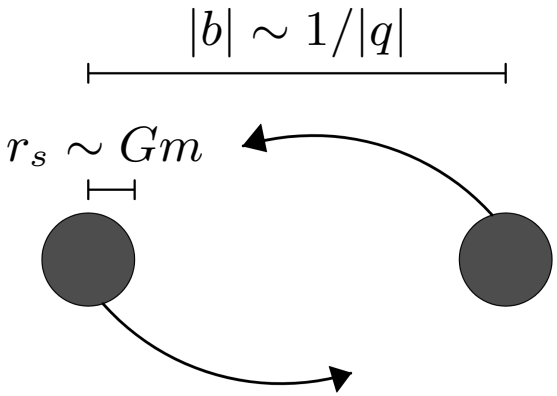
CY $(2L-2)$-fold

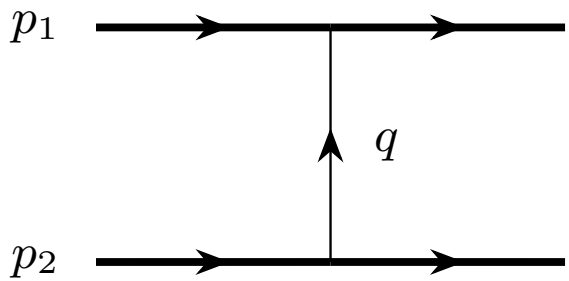
[Bourjaily, McLeod, von Hippel, Wilhelm (2018)], [Lairez, Vanhove (2022)]

$\hookrightarrow$ Are there also non-trivial geometries in gravitational waves? ✓

[Frellesvig, **RM**, Wilhelm (2023)], [Klemm, Nega, Sauer, Plefka (2024)]

EFT and hierarchy of scales

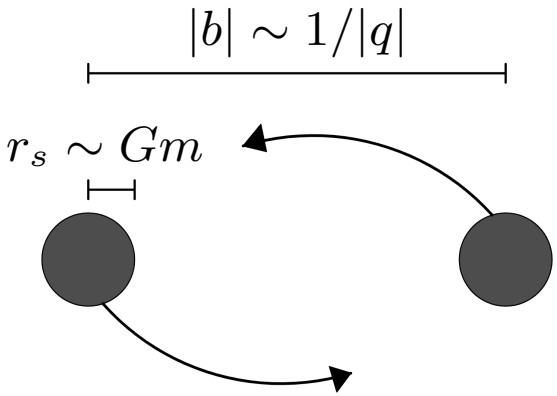


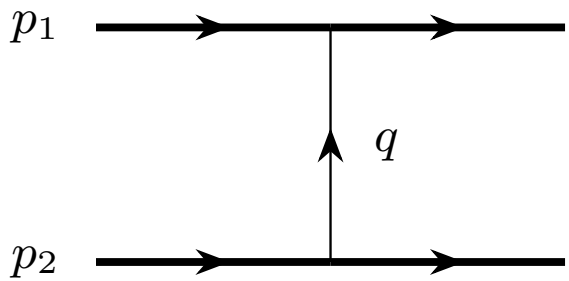


- Point-particle limit: $r_s \ll |b| \Longrightarrow$ **black holes = massive scalars**

[Cheung, Rothstein, Solon (2018)], . . . , [see talks by Benjamin, Christoph, Radu, . . .]

EFT and hierarchy of scales



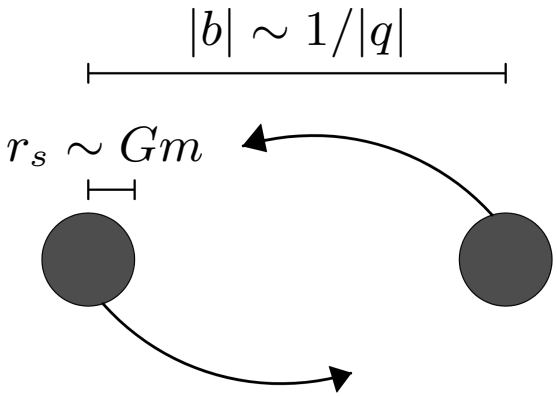


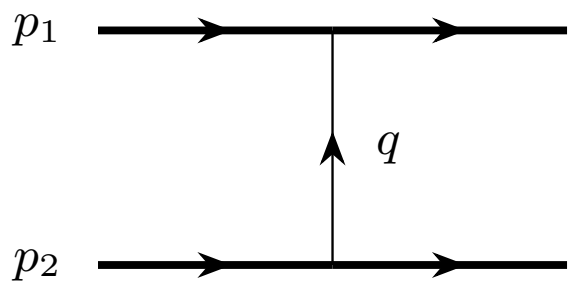
- Point-particle limit: $r_s \ll |b| \Longrightarrow$ **black holes = massive scalars**
- Classical limit: $J \gg \hbar \Longrightarrow s, |u|, m_1, m_2 \sim J^2|t| \gg |t| = |q|^2$

$\hbar$ scaling $\rightarrow$ $|q|$ scaling (soft-$|q|$ expansion)

[Cheung, Rothstein, Solon (2018)], . . . , [see talks by Benjamin, Christoph, Radu, . . .]

EFT and hierarchy of scales





- Point-particle limit: $r_s \ll |b| \Longrightarrow$ **black holes = massive scalars**
- Classical limit: $J \gg \hbar \Longrightarrow s, |u|, m_1, m_2 \sim J^2|t| \gg |t| = |q|^2$

$$\hbar \text{ scaling} \to |q| \text{ scaling (soft-}|q|\text{ expansion)}$$

- Dependence on one dimensionless variable!

$$y = \frac{1+x^2}{2x} = \frac{p_1 \cdot p_2}{m_1 m_2} + \mathcal{O}(|q|^2)$$

To rationalize $\sqrt{y^2-1}$

[Cheung, Rothstein, Solon (2018)], . . . , [see talks by Benjamin, Christoph, Radu, . . .]

Outline

1. Geometries in Feynman integrals
2. How to detect non-trivial geometries
3. Results: Classification of PM geometries
4. Conclusions and outlook

Detecting geometries: Differential equations

Differential equations for master integrals $\vec{I} = (I_1, \ldots, I_n)$ [Kotikov (1991)], [Henn (2013)], ...

$$\partial_x \vec{I} = A\vec{I}$$

Detecting geometries: Differential equations

Differential equations for master integrals $\vec{I} = (I_1, \ldots, I_n)$ [Kotikov (1991)], [Henn (2013)], ...

$$\partial_x \vec{I} = A\vec{I}$$

$\hookrightarrow$ Higher-order differential equation

$$\mathcal{L}_n I_i = \text{inhomogeneity} \longleftarrow \text{(subsectors)}$$

with the Picard-Fuchs operator

$$\mathcal{L}_n = \frac{d^n}{dx^n} + c_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + c_1(x)\frac{d}{dx} + c_0(x)$$

Detecting geometries: Differential equations

Differential equations for master integrals $\vec{I} = (I_1, \ldots, I_n)$ [Kotikov (1991)], [Henn (2013)], ...

$$\partial_x \vec{I} = A\vec{I}$$

$\hookrightarrow$ Higher-order differential equation

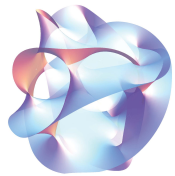
$$\mathcal{L}_n I_i = \text{inhomogeneity} \longleftarrow \text{(subsectors)}$$

with the Picard-Fuchs operator

$$\mathcal{L}_n = \frac{d^n}{dx^n} + c_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + c_1(x)\frac{d}{dx} + c_0(x)$$

Detecting geometries:

- $\mathcal{L}_n$ factorizes into $\mathcal{L}_1$'s $\Longrightarrow$ polylogarithmic (if inhomogeneity is, too)
- $\mathcal{L}_n$ factorizes into $\mathcal{L}_{(r>1)}$ $\Longrightarrow$ beyond polylogarithms (normally a CY $(r-1)$-fold)



Detecting geometries: Differential equations

Differential equations for master integrals $\vec{I} = (I_1, \ldots, I_n)$ [Kotikov (1991)], [Henn (2013)], ...

$$\partial_x \vec{I} = A\vec{I}$$

$\hookrightarrow$ Higher-order differential equation

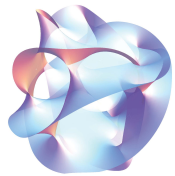
$$\mathcal{L}_n I_i = \text{inhomogeneity} \longleftarrow \text{(subsectors)}$$

with the Picard-Fuchs operator

$$\mathcal{L}_n = \frac{d^n}{dx^n} + c_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + c_1(x)\frac{d}{dx} + c_0(x)$$

Detecting geometries:

- $\mathcal{L}_n$ factorizes into $\mathcal{L}_1$'s $\Longrightarrow$ polylogarithmic (if inhomogeneity is, too)
- $\mathcal{L}_n$ factorizes into $\mathcal{L}_{(r>1)}$ $\Longrightarrow$ beyond polylogarithms (normally a CY $(r-1)$-fold)



Disadvantage (usually): We need to solve IBPs and calculate the DEs!

Detecting geometries (reloaded): Leading singularities

Leading singularity $\sim$ maximally iterated discontinuity [Cachazo (2008)],...

Detecting geometries (reloaded): Leading singularities

Leading singularity $\sim$ maximally iterated discontinuity [Cachazo (2008)],...

Previous example with $y^2 = P(x_2, \ldots, x_n)$:

$$I = \int_0^1 dx_2 \ldots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (+ \to -) \right)$$

$$\mathrm{LS}(I) = \int_0^1 dx_2 \ldots dx_n \frac{1}{2y} \longleftarrow \text{Precisely captures the geometry!}$$

Detecting geometries (reloaded): Leading singularities

Leading singularity $\sim$ maximally iterated discontinuity [Cachazo (2008)],...

Previous example with $y^2 = P(x_2, \ldots, x_n)$:

$$I = \int_0^1 dx_2 \ldots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (+ \to -) \right)$$

$$\mathrm{LS}(I) = \int_0^1 dx_2 \ldots dx_n \frac{1}{2y} \longleftarrow \text{Precisely captures the geometry!}$$

$\Longrightarrow$ Replace propagators by on-shell delta functions: $\dfrac{i}{Q_i^2 - m_i^2} \to \delta(Q_i^2 - m_i^2)$

Detecting geometries (reloaded): Leading singularities

Leading singularity $\sim$ maximally iterated discontinuity [Cachazo (2008)],...

Previous example with $y^2 = P(x_2, \ldots, x_n)$:

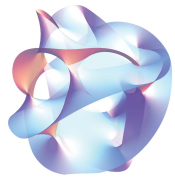
$$I = \int_0^1 dx_2 \ldots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (+ \to -) \right)$$

$$\mathrm{LS}(I) = \int_0^1 dx_2 \ldots dx_n \frac{1}{2y} \longleftarrow \text{Precisely captures the geometry!}$$

$\Longrightarrow$ Replace propagators by on-shell delta functions: $\dfrac{i}{Q_i^2 - m_i^2} \to \delta(Q_i^2 - m_i^2)$

Detecting geometries:

- LS of Feynman integral (& subsectors) is algebraic $\Longrightarrow$ polylogarithmic
- Otherwise $\Longrightarrow$ beyond polylogarithms



Detecting geometries (reloaded): Leading singularities

Leading singularity $\sim$ maximally iterated discontinuity [Cachazo (2008)],...

Previous example with $y^2 = P(x_2, \ldots, x_n)$:

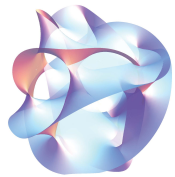
$$I = \int_0^1 dx_2 \ldots dx_n \left(\frac{\log(1 - 1/y)}{2y} - (+ \to -) \right)$$

$$\mathrm{LS}(I) = \int_0^1 dx_2 \ldots dx_n \frac{1}{2y} \longleftarrow \text{Precisely captures the geometry!}$$

$\Longrightarrow$ Replace propagators by on-shell delta functions: $\dfrac{i}{Q_i^2 - m_i^2} \to \delta(Q_i^2 - m_i^2)$

Detecting geometries:

- LS of Feynman integral (& subsectors) is algebraic $\Longrightarrow$ polylogarithmic
- Otherwise $\Longrightarrow$ beyond polylogarithms



Advantage: Less integrals $\Longrightarrow$ Easier to calculate! ✓

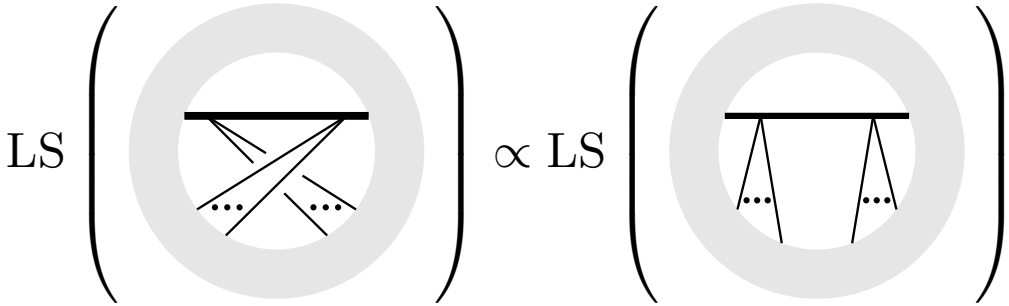
$\hookrightarrow$ Baikov representation [Baikov (1997)], [Frellesvig, Papadopoulos (2017)]

Outline

1. Geometries in Feynman integrals
2. How to detect non-trivial geometries
3. Results: Classification of PM geometries
4. Conclusions and outlook

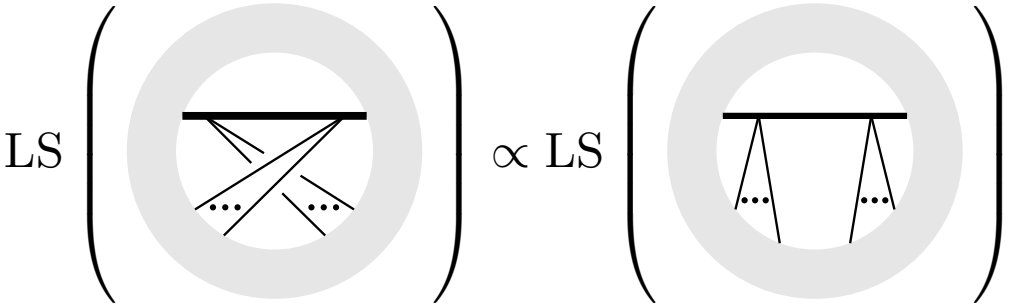
Relating geometries through Baikov

- Vertices at matter lines become orderless: [Frellesvig, **RM**, Wilhelm (2024)]

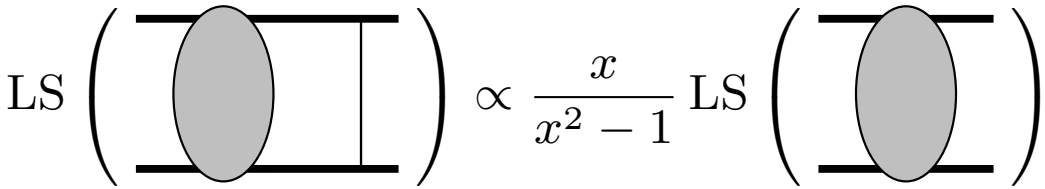


Relating geometries through Baikov

- Vertices at matter lines become orderless: [Frellesvig, **RM**, Wilhelm (2024)]

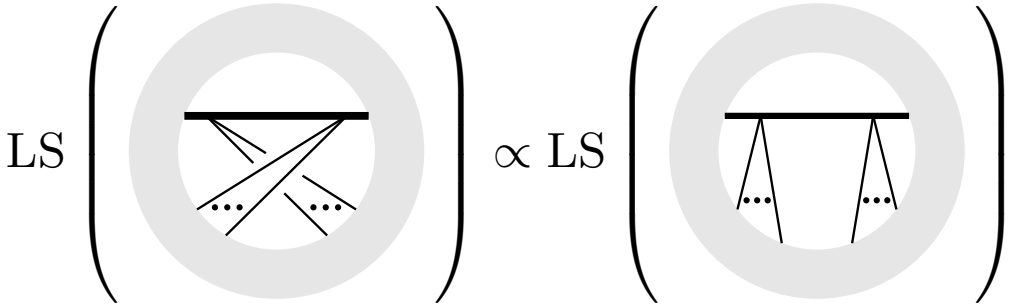


- We can reduce superclassical diagrams:

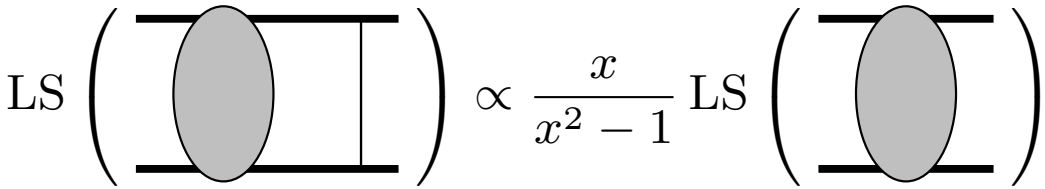


Relating geometries through Baikov

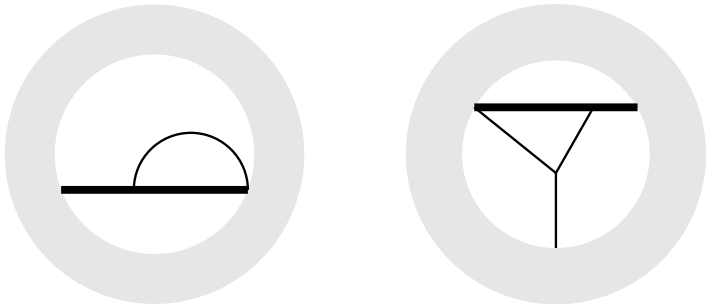
- Vertices at matter lines become orderless: [Frellesvig, **RM**, Wilhelm (2024)]



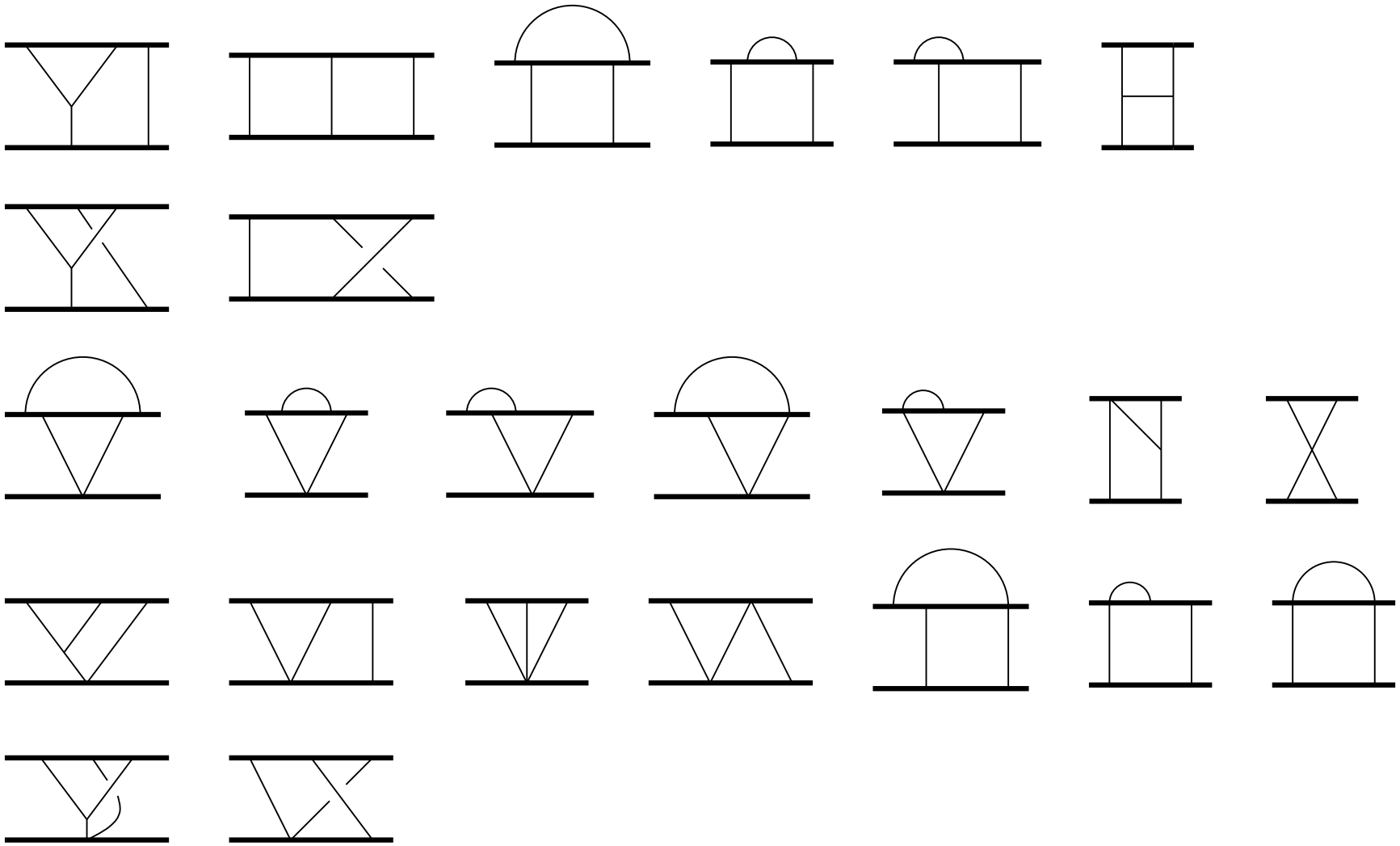
- We can reduce superclassical diagrams:



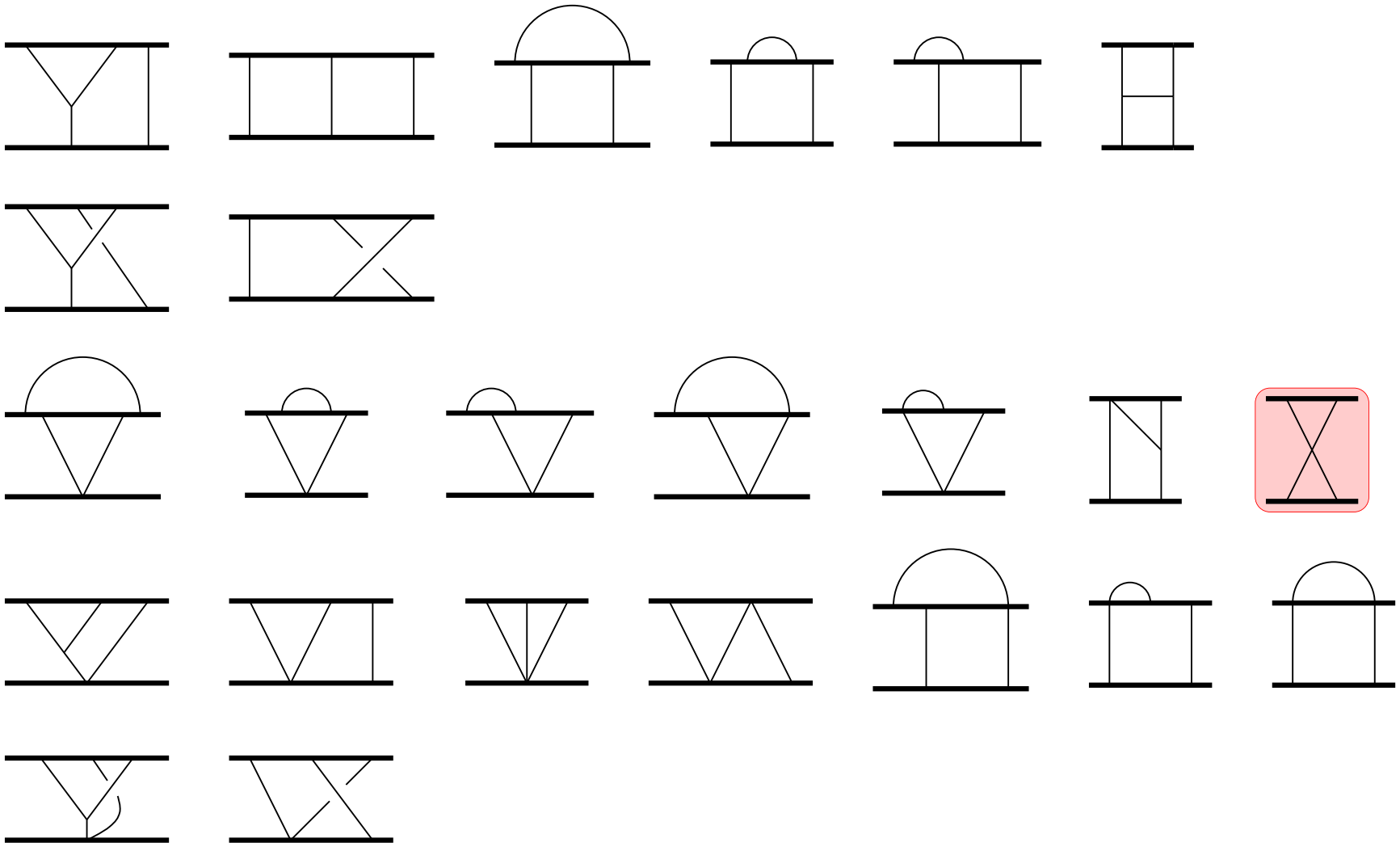
- Certain diagrams have zero master integrals:



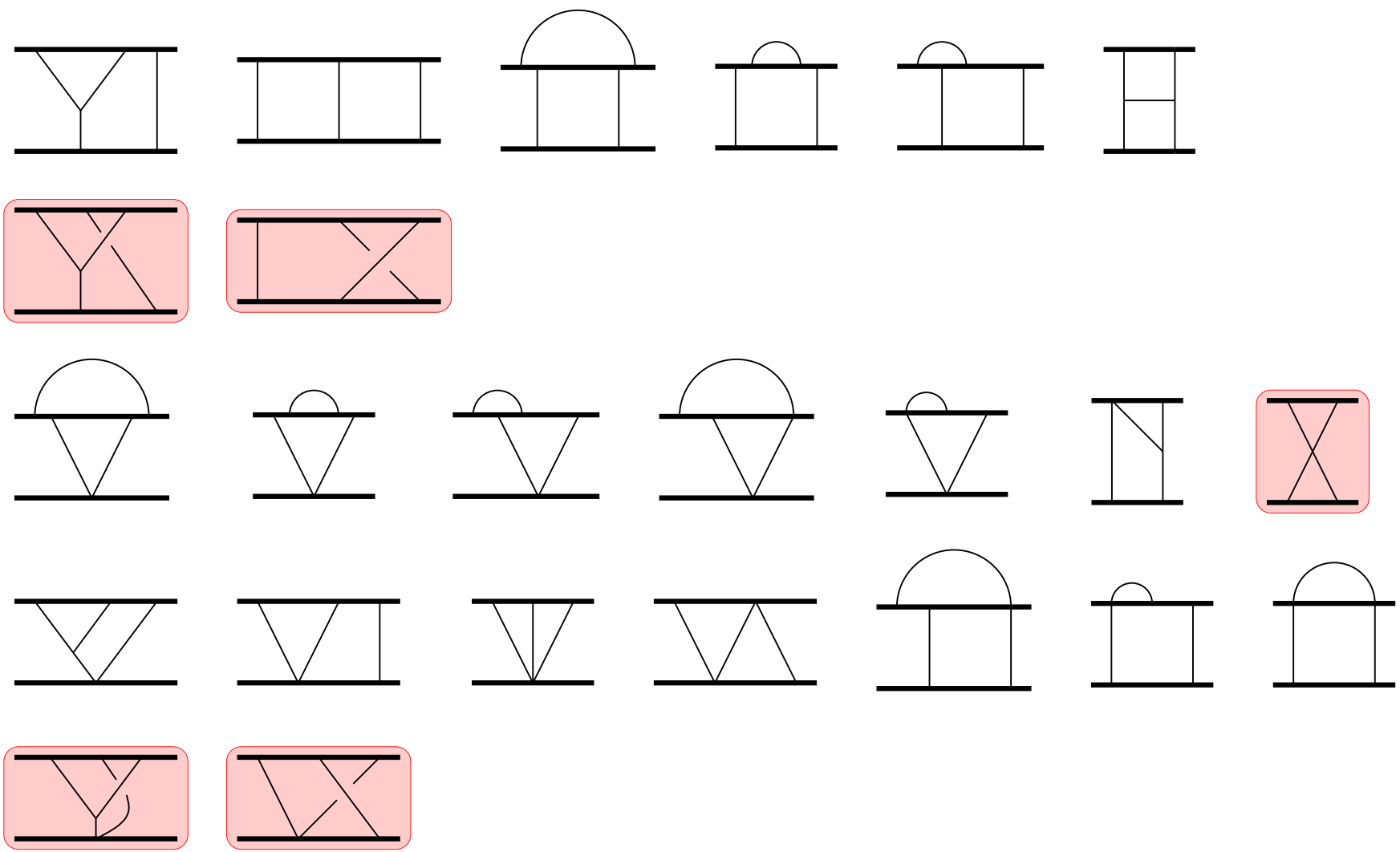
Simplifying relations at two loops



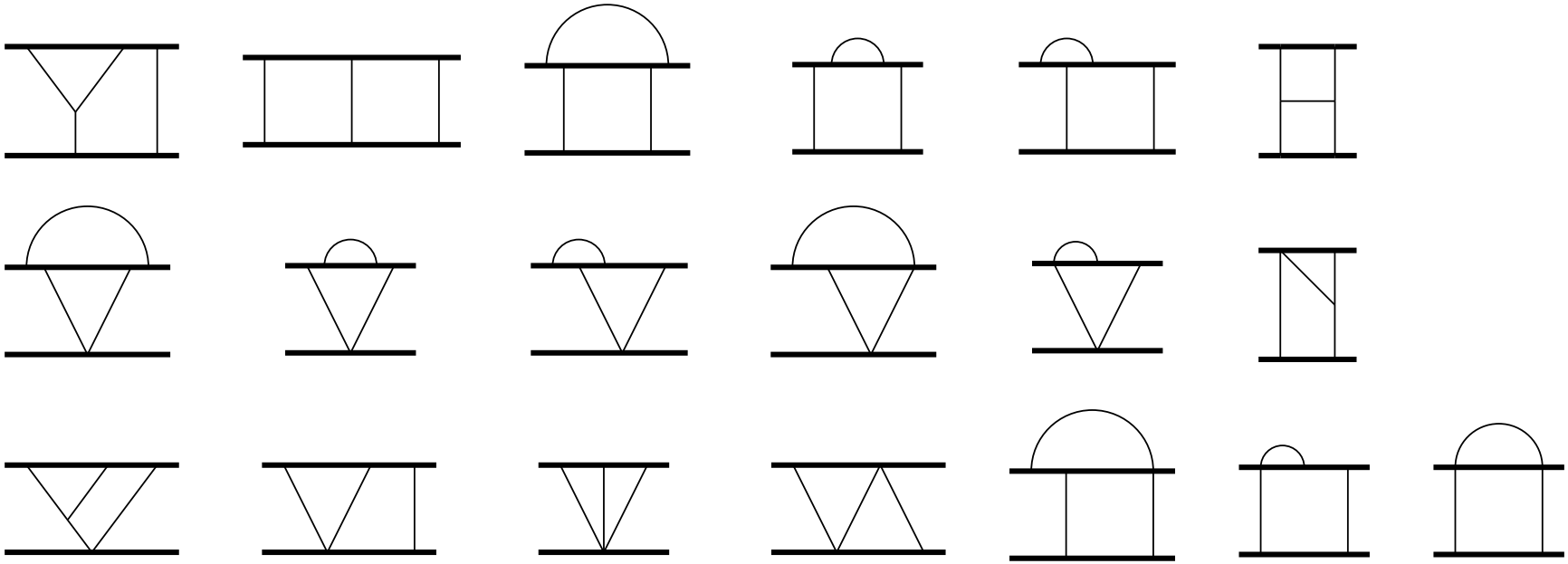
Simplifying relations at two loops



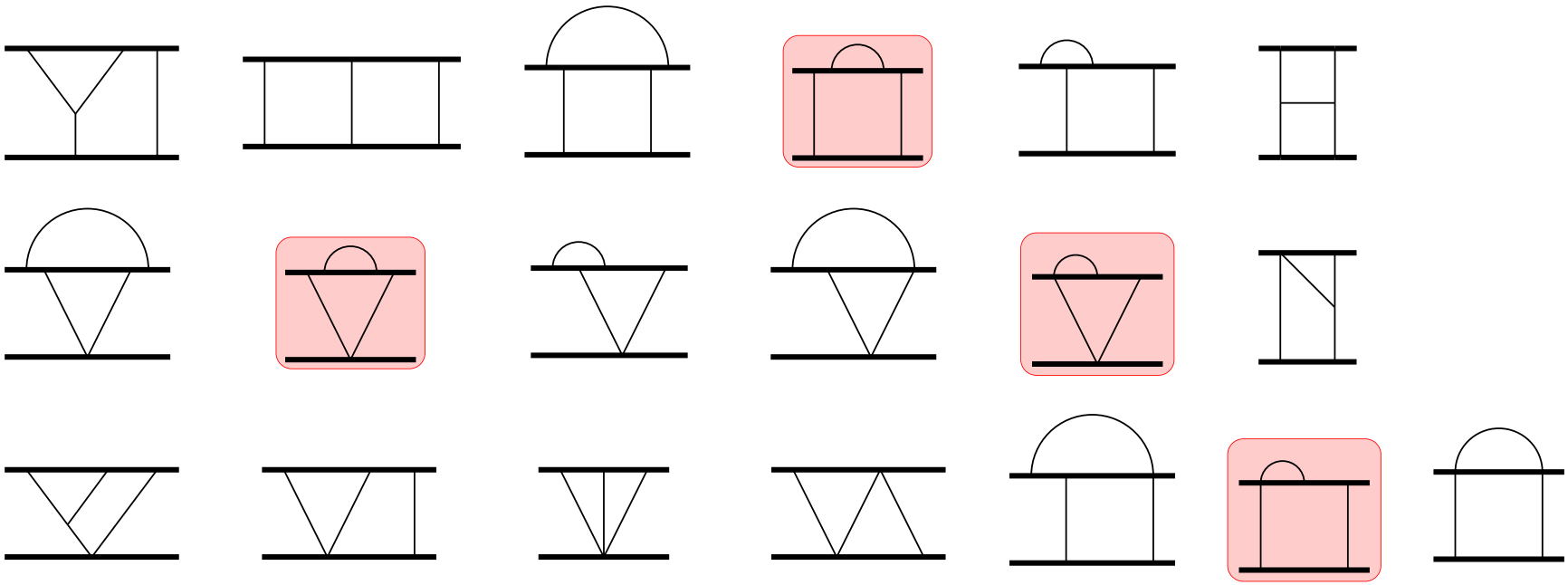
Simplifying relations at two loops



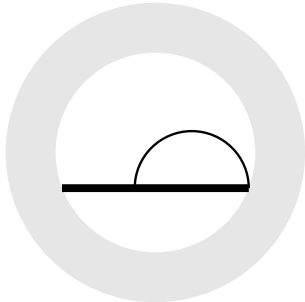
Simplifying relations at two loops



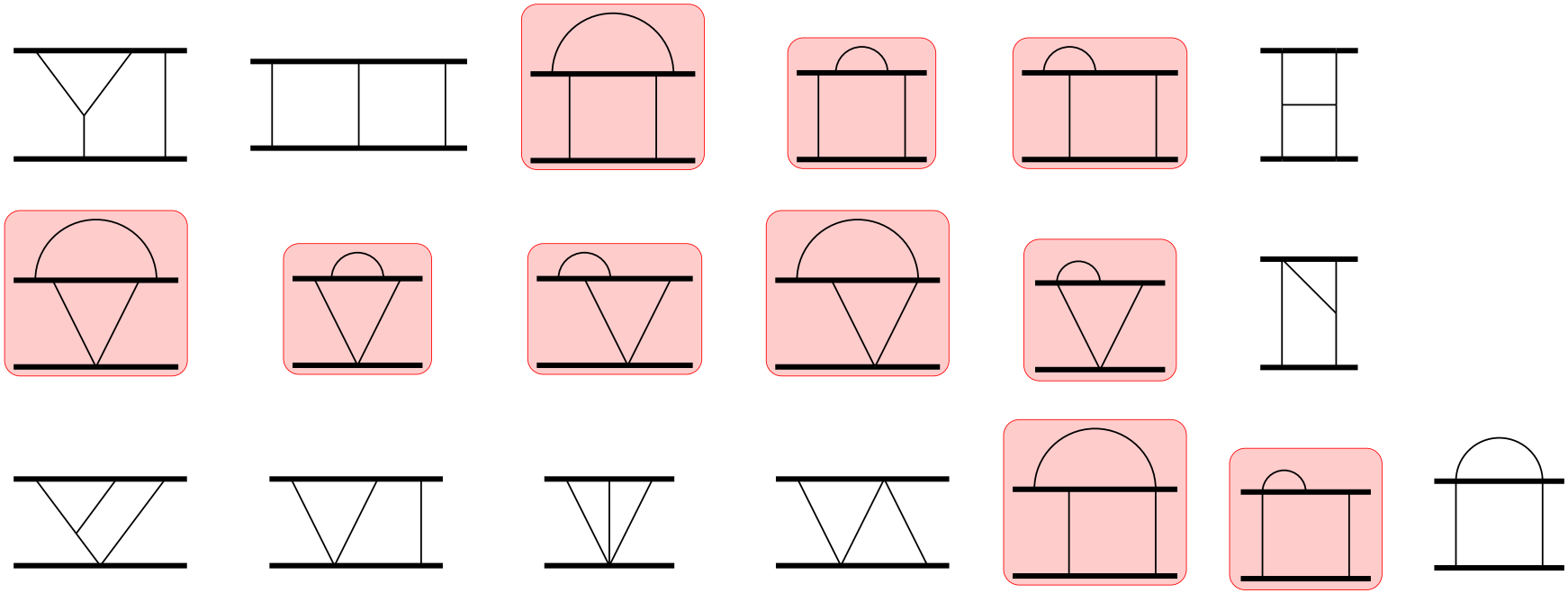
Simplifying relations at two loops



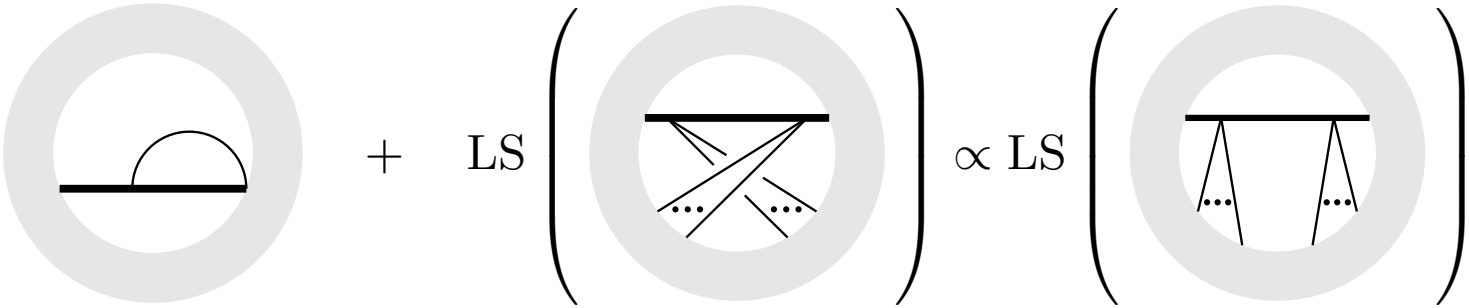
$\Longrightarrow$ We can remove bubbles with zero master integrals:



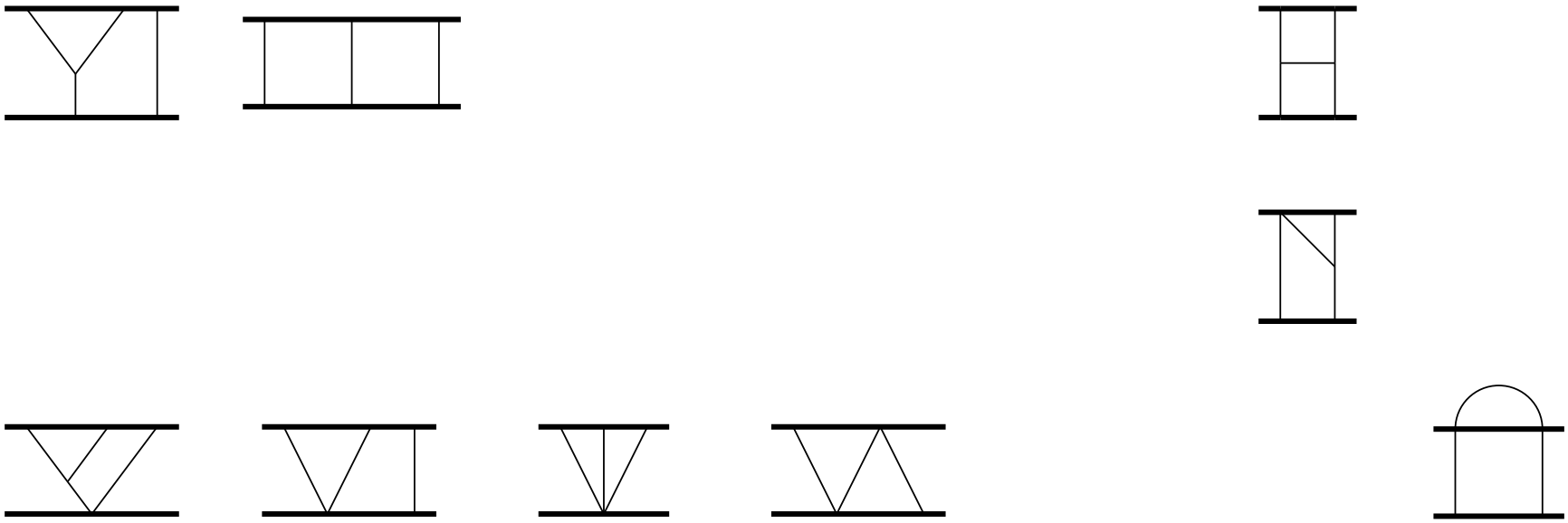
Simplifying relations at two loops



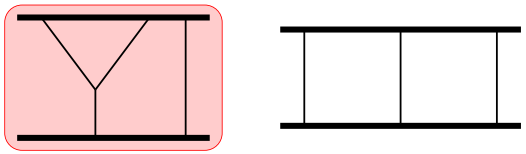
$\Longrightarrow$ We can remove more bubbles with zero master integrals:

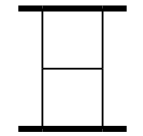


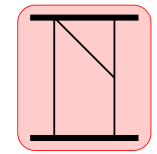
Simplifying relations at two loops

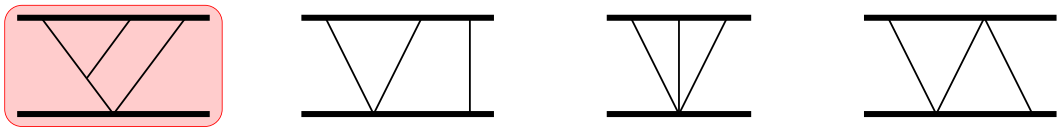


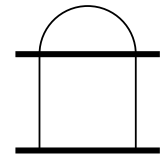
Simplifying relations at two loops



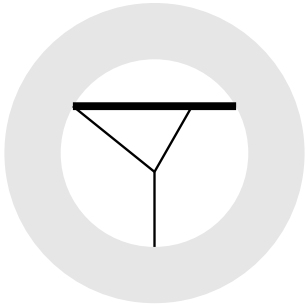




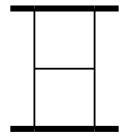


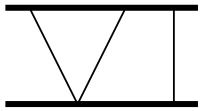


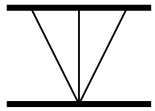
$\Longrightarrow$ We can remove triangles with zero master integrals:

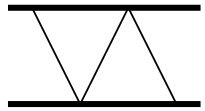


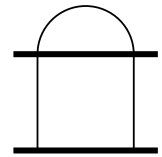
Simplifying relations at two loops



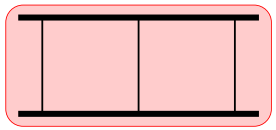
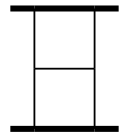
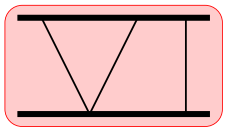
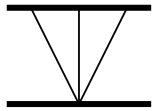
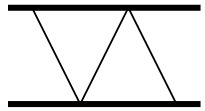
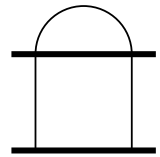








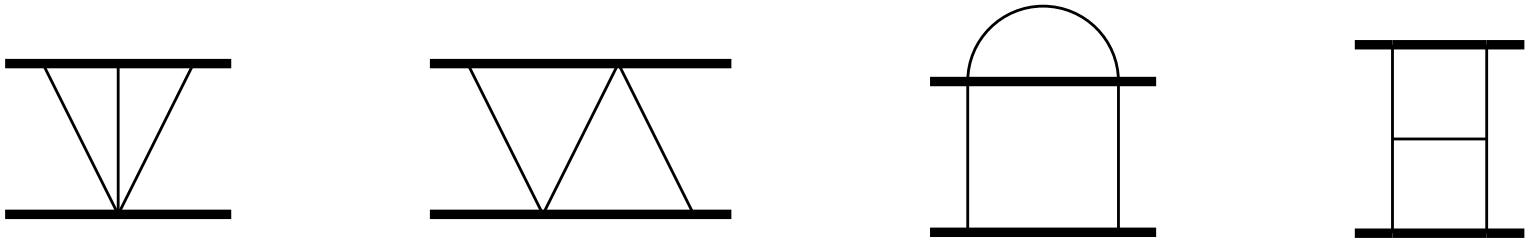
Simplifying relations at two loops

$\Longrightarrow$ We can reduce superclassical diagrams:

$$\text{LS}\left(\quad\right) \propto \frac{x}{x^2-1}\,\text{LS}\left(\quad\right)$$

Simplifying relations at two loops



↪ Only four relevant diagrams at two loops!

Full classification up to four loops

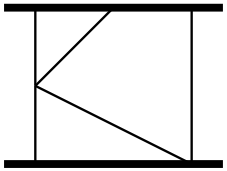
- These simplifying relations allow for a full classification!

L	1	2	3	4
Initial number	2	23	531	16596
Reduction of superclassical diagrams	1	12	271	8335
Reduction of bubbles	1	7	100	1854
Reduction of triangles	1	4	33	476
Unraveling of matter propagators	1	4	15	79

[Frellesvig, **RM**, Wilhelm (2024)]

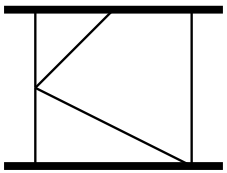
[Brammer, Frellesvig, **RM**, Wilhelm (to appear)]

Example: A K3 surface at three loops



[Bern, Parra-Martinez, Robian, Ruf, Shen, Solon, Zeng (2021)]
[Dlapa, Kälin, Liu, Porto (2021)]

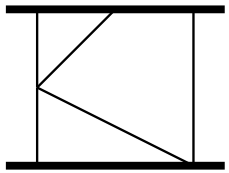
Example: A K3 surface at three loops



[Bern, Parra-Martinez, Robian, Ruf, Shen, Solon, Zeng (2021)]
[Dlapa, Kälin, Liu, Porto (2021)]

- Loop-by-loop Baikov: [Frellesvig, **RM**, Wilhelm (2024)]

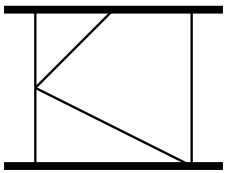
$$\text{LS}\left(\quad\right) \propto x \int \frac{dz_1 dz_2}{\sqrt{P_6(z_1, z_2)}}$$



with $\quad P_6(z_1, z_2) = z_2^2(z_1^2 - 1)^2(x^2 - 1)^2 - 4x^2 z_1^2(z_2^2 + 1)^2$

$\hookrightarrow$ $y^2 = P_6(z_1, z_2)$ defines a K3 surface!

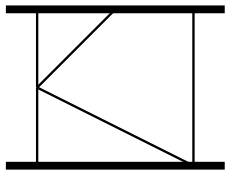
Example: A K3 surface at three loops



[Bern, Parra-Martinez, Robian, Ruf, Shen, Solon, Zeng (2021)]
[Dlapa, Kälin, Liu, Porto (2021)]

- Loop-by-loop Baikov: [Frellesvig, **RM**, Wilhelm (2024)]

$$\text{LS}\left(\quad\right) \propto x \int \frac{dz_1 dz_2}{\sqrt{P_6(z_1,z_2)}}$$



with $\quad P_6(z_1,z_2) = z_2^2(z_1^2-1)^2(x^2-1)^2 - 4x^2 z_1^2(z_2^2+1)^2$

$\hookrightarrow$ $y^2 = P_6(z_1,z_2)$ defines a K3 surface!

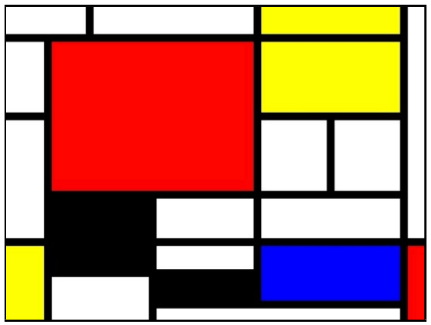
- Picard-Fuchs operator in $D=4$: [Dlapa, Henn, Wagner (2022)]

$$\mathcal{L}_3 = \frac{\partial^3}{\partial x^3} - \frac{6x}{1-x^2}\frac{\partial^2}{\partial x^2} + \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{\partial}{\partial x} - \frac{1+x^2}{x^3(1-x^2)}$$

$\hookrightarrow$ $\mathcal{L}_3$ is symmetric square of $\mathcal{L}_2 \Longrightarrow$ Products of elliptic integrals
[Joyce (1972)], [Verrill (1996)], [Doran (1998)]

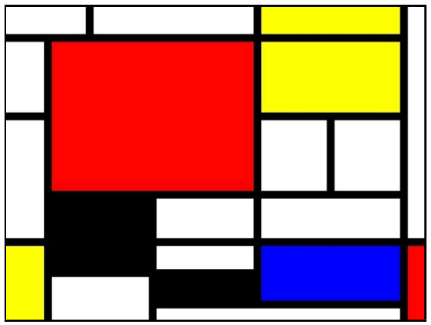
Classification at four loops

- Only Mondrian family of diagrams (+ non-planar) is interesting:
  [Brammer, Frellesvig, **RM**, Wilhelm (forthcoming work)]



Classification at four loops

- Only Mondrian family of diagrams (+ non-planar) is interesting:
  [Brammer, Frellesvig, **RM**, Wilhelm (forthcoming work)]



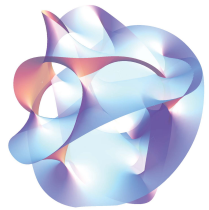
- Loop-by-loop Baikov: [Frellesvig, **RM**, Wilhelm (2023)]

$$\text{LS}\left(\quad\right) \propto x \int \frac{dz_1 dz_2 dz_3}{\sqrt{P_8(z_1,z_2,z_3)}}$$

with $P_8(z_1,z_2,z_3) = (1+z_1)^2\,(1+z_2)^2\,(z_1 z_2+z_3^2)^2\,(1-x^2)^2$

$$+64\,x^2 z_1^2\,z_2^2\,(1+z_1+z_2)\,z_3^2$$

$\hookrightarrow$ $y^2 = P_8(z_1,z_2,z_3)$ defines Calabi-Yau three-fold!



A Calabi-Yau three-fold at 5PM order

Cross-check: Picard-Fuchs operator [Frellesvig, **RM**, Wilhelm (2023)]

$$\mathcal{L}_4 = \frac{\partial^4}{\partial x^4} + \frac{2-16x^2-10x^4}{x(1-x^4)}\frac{\partial^3}{\partial x^3} + \frac{1-28x^2+46x^4+68x^6+25x^8}{x^2(1-x^4)^2}\frac{\partial^2}{\partial x^2}$$
$$- \frac{1+11x^2-54x^4+22x^6+37x^8+15x^{10}}{x^3(1-x^2)^3(1+x^2)^2}\frac{\partial}{\partial x}$$
$$+ \frac{1+3x^2+20x^4+3x^6+x^8}{x^4(1-x^4)^2}$$

$\Longrightarrow$ Does not factor! ✓

A Calabi-Yau three-fold at 5PM order

Cross-check: Picard-Fuchs operator [Frellesvig, **RM**, Wilhelm (2023)]

$$\mathcal{L}_4 = \frac{\partial^4}{\partial x^4} + \frac{2-16x^2-10x^4}{x(1-x^4)}\frac{\partial^3}{\partial x^3} + \frac{1-28x^2+46x^4+68x^6+25x^8}{x^2(1-x^4)^2}\frac{\partial^2}{\partial x^2} - \frac{1+11x^2-54x^4+22x^6+37x^8+15x^{10}}{x^3(1-x^2)^3(1+x^2)^2}\frac{\partial}{\partial x} + \frac{1+3x^2+20x^4+3x^6+x^8}{x^4(1-x^4)^2}$$

$\Longrightarrow$ Does not factor! ✓

Properties of $\mathcal{L}_4$: [Frellesvig, **RM**, Pögel, Weinzierl, Wilhelm (to appear)]

- essentially self-adjoined ✓
- $x = 0$ is maximal unipotent monodromy (MUM) point with integer solution ✓
- ... ✓

$\Longrightarrow$ Calabi-Yau operator $\Longrightarrow$ New functions!

A Calabi-Yau three-fold at 5PM order

Cross-check: Picard-Fuchs operator [Frellesvig, **RM**, Wilhelm (2023)]

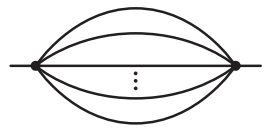
$$\mathcal{L}_4 = \frac{\partial^4}{\partial x^4} + \frac{2-16x^2-10x^4}{x(1-x^4)}\frac{\partial^3}{\partial x^3} + \frac{1-28x^2+46x^4+68x^6+25x^8}{x^2(1-x^4)^2}\frac{\partial^2}{\partial x^2} - \frac{1+11x^2-54x^4+22x^6+37x^8+15x^{10}}{x^3(1-x^2)^3(1+x^2)^2}\frac{\partial}{\partial x} + \frac{1+3x^2+20x^4+3x^6+x^8}{x^4(1-x^4)^2}$$

$\Longrightarrow$ Does not factor! ✓

Properties of $\mathcal{L}_4$: [Frellesvig, **RM**, Pögel, Weinzierl, Wilhelm (to appear)]

- essentially self-adjoined ✓
- $x = 0$ is maximal unipotent monodromy (MUM) point with integer solution ✓
- ... ✓

$\Longrightarrow$ Calabi-Yau operator $\Longrightarrow$ New functions!



[Pögel, Wang, Weinzierl (2022)]

$\hookrightarrow$ We can calculate a series expansion

Outline

1. Geometries in Feynman integrals
2. How to detect non-trivial geometries
3. Results: Classification of PM geometries
4. Conclusions and outlook

Conclusions and outlook

Conclusions:

1. Non-trivial geometries appear in Feynman integrals for both collider physics and gravity
2. Loop-by-loop Baikov + DEs = powerful tools for classifying geometries in the PM expansion
3. Identified a CY three-fold at four loops

Conclusions and outlook

Conclusions:

1. Non-trivial geometries appear in Feynman integrals for both collider physics and gravity
2. Loop-by-loop Baikov + DEs = powerful tools for classifying geometries in the PM expansion
3. Identified a CY three-fold at four loops

Future work:

1. Calculate the CY integral
2. Classification at higher loops?
3. Waveform?

Conclusions and outlook

Conclusions:

1. Non-trivial geometries appear in Feynman integrals for both collider physics and gravity
2. Loop-by-loop Baikov + DEs = powerful tools for classifying geometries in the PM expansion
3. Identified a CY three-fold at four loops

Future work:

1. Calculate the CY integral
2. Classification at higher loops?
3. Waveform?

Thank you!

Backup: Classical limit (I)

Feynman rules:

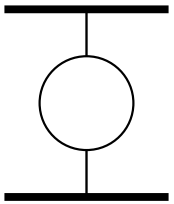
$$\longrightarrow = \frac{1}{k^2} \sim |q|^{-2}, \qquad \longrightarrow\!\!\!

Backup: Classical limit (II)

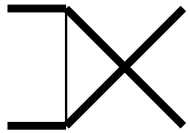
Feynman diagrams at L loops $\sim G^{L+1}$:

- $|q|^{>L-2}$ **quantum** ✗
- $|q|^{L-2}$ **classical** ✓
- $|q|^{<L-2}$ **superclassical** (iterations) ✗

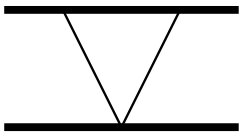
Examples (2PM $\sim |q|^{-1}G^2$):



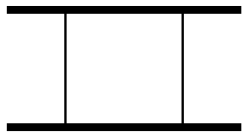
$\sim |q|^0 G^2$ ✗

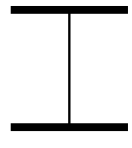


$\longleftarrow$ BH contact term ✗



$\sim |q|^{-1}G^2$ ✓

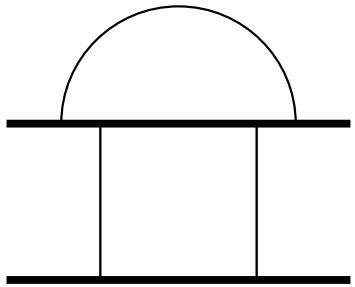


$\longleftarrow$ Iteration $\sim$  ✗

[Neill, Rothstein (2013)], [Bern, Cheung, Roiban, Shen, Solon, Zeng (2019)], ...

Backup: Rad. reaction and cancellation of geometries

We can equally study conservative and dissipative dynamics $\Longrightarrow$ We also include radiation reaction



Assumption: Absence of total cancellations in the sum of Feynman diagrams

- Unlike $\mathcal{N} = 4$ SYM theory
- Supported by three loops [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng (2021)], [Dlapa, Kälin, Liu, Porto (2021)]
- No elliptic integrals at conservative four loops at 1SF [Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch (2024)]

$\Longrightarrow$ Individual Feynman integrals indicative of final amplitude

Backup: Baikov representation

Feynman integral with n_{int} propagators

$$I = \int d^D k_1 \cdots d^D k_L \prod_{i=1}^{n_{\text{int}}} \frac{1}{Q_i^2 - m_i^2}$$

Baikov representation: change of variables to $z_i = Q_i^2 - m_i^2$ and ISPs [Baikov (1997)]

$$I = \mathcal{C}\, \mathcal{G}^{\frac{-D+E+1}{2}} \int \frac{d^{n_{\text{int}}+n_{\text{ISP}}} z}{z_1 \cdots z_{n_{\text{int}}}} \mathcal{B}(z)^{\frac{D-L-E-1}{2}}$$

- $E = \#$ independent external momenta
- Baikov polynomial $\mathcal{B}(z) = \det G(k_1, \ldots, k_L, p_1, \ldots, p_E)$
  $\mathcal{G} = \det G(p_1, \ldots, p_E)$

$\Longrightarrow$ Straightforward leading singularity via residues:

$$\text{LS}(I) \propto \mathcal{G}^{\frac{-D+E+1}{2}} \int d^{n_{\text{ISP}}} z\; \mathcal{B}(\underbrace{0, \ldots, 0}_{n_{\text{int}}}, z)^{\frac{D-L-E-1}{2}}$$

Variant: Loop-by-loop Baikov representation [Frellesvig, Papadopoulos (2017)]