Calabi-Yau in the Sky? Black hole scattering at 4 loops

Work with Mathias Driesse, Gustav Jakobsen, Albrecht Klemm, Gustav Mogull, Christoph Nega, Jan Plefka, Johann Usovitsch, Yingxuan Xu

4 loop:

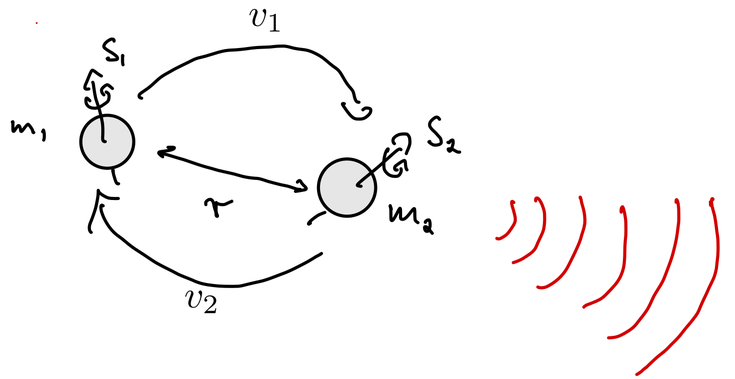
2409.***** Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, **BS**, Usovitsch

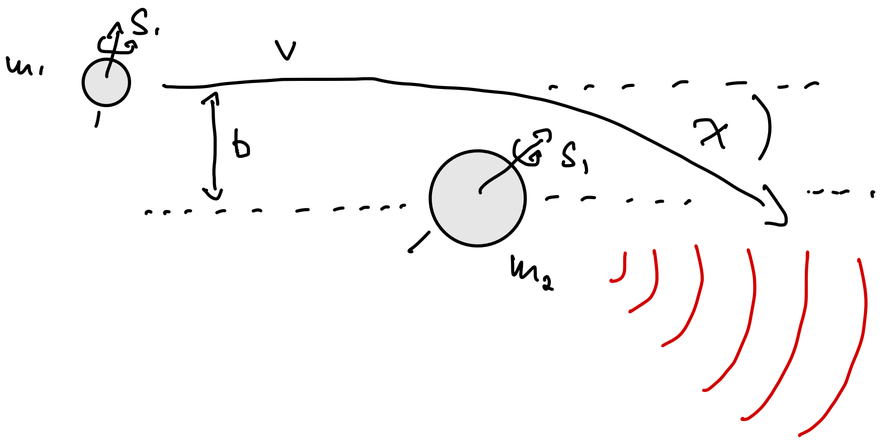
2403.07781 PRL 132 (2024) 24, Driesse, Jakobsen, Mogull, Plefka, **BS**, Usovitsch
2401.07899 PRD 109 (2024) 12, Klemm, Nega, **BS**, Plefka

3 loop:
2308.11514 PRL 131 (2023) 24, Jakobsen, Mogull, Plefka, **BS**
2306.01714 PRL 131 (2023) 15, Jakobsen, Mogull, Plefka, **BS**, Xu

The general relativistic 2-body problem

Bound Case: Post-Newtonian expansion in v and G





Unbound Case: Post-Minkowskian weak field expansion in G

➡Mapping back to bound case possible [see Christophs talk]

➡Generation of waveforms via e.g. EOB approach

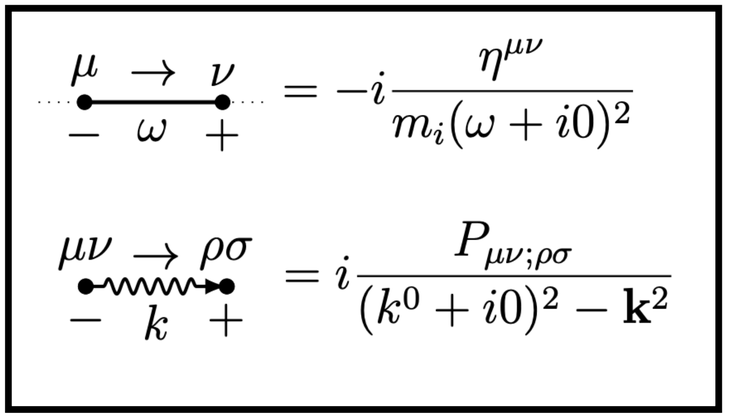
Worldline Quantum Field Theory

Mogull,Plefka,Steinhoff [2010.02865]

$$S = -\sum_{i=1}^{2} \int d\tau_i \sqrt{g_{\mu\nu}\dot{x}_i^\mu(\tau_i)\dot{x}_i^\nu(\tau_i)} + \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_{\text{g.f.}}$$

Point particle approximation

Bulk gravity & gauge fixing

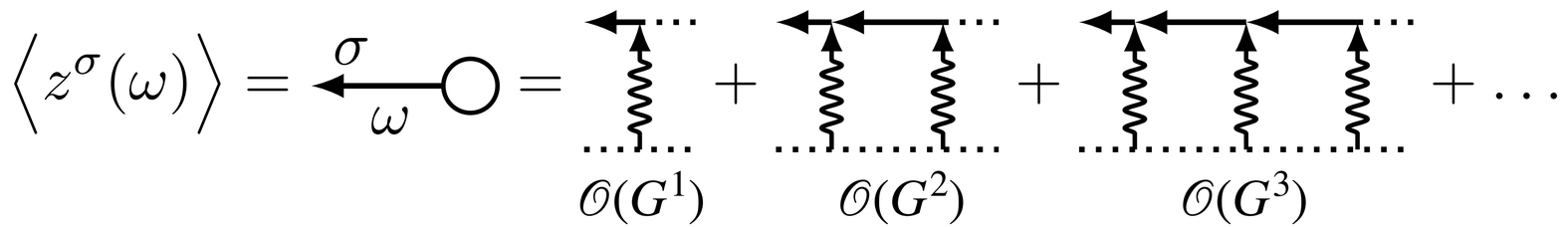


WQFT: Quantization of deflection and perturbation of flat space

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu\tau + z_i^\mu(\tau) \qquad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\,h_{\mu\nu}$$

Tree-level one-point functions solve **classical** equations of motion.
Very easy classical limit!

$$\Delta p_1^\mu = \lim_{\omega\to 0} \omega^2 < z_1^\mu(\omega) >$$

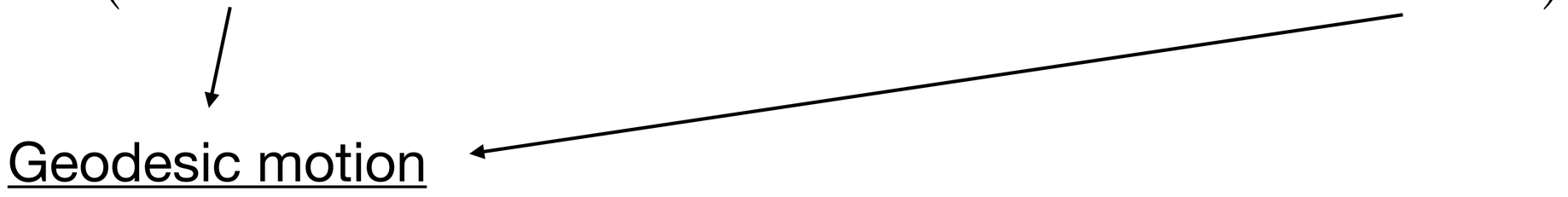


Scattering at 5PM

$$\Delta p_i^{(5)\mu} = m_1 m_2 \left(m_1^4 \Delta p_{0SF}^{(5)\mu} + m_1^3 m_2 \Delta p_{1SF}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{2SF}^{(5)\mu} + m_1 m_2^3 \Delta \bar{p}_{1SF}^{(5)\mu} + m_2^4 \Delta \bar{p}_{0SF}^{(5)\mu} \right)$$

Scattering at 5PM

$$\Delta p_i^{(5)\mu} = m_1 m_2 \left(m_1^4 \Delta p_{0SF}^{(5)\mu} + m_1^3 m_2 \Delta p_{1SF}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{2SF}^{(5)\mu} + m_1 m_2^3 \Delta \bar{p}_{1SF}^{(5)\mu} + m_2^4 \Delta \bar{p}_{0SF}^{(5)\mu} \right)$$



Scattering at 5PM

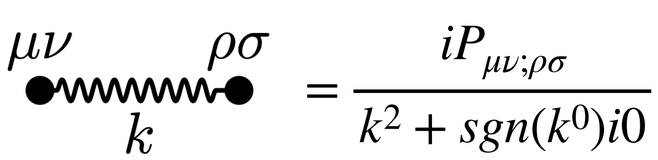
$$\nu = m_1 m_2/(m_1+m_2)$$

$$\Delta p_i^{(5)\mu} = m_1 m_2 \Big(\underbrace{m_1^4 \Delta p_{0SF}^{(5)\mu}}_{\mathcal{O}(\nu^0)} + \underbrace{m_1^3 m_2 \Delta p_{1SF}^{(5)\mu}}_{\mathcal{O}(\nu)} + \underbrace{m_1^2 m_2^2 \Delta p_{2SF}^{(5)\mu}}_{\mathcal{O}(\nu^2)} + \underbrace{m_1 m_2^3 \Delta \bar{p}_{1SF}^{(5)\mu}}_{\mathcal{O}(\nu)} + \underbrace{m_2^4 \Delta \bar{p}_{0SF}^{(5)\mu}}_{\mathcal{O}(\nu^0)} \Big)$$

Scattering at 5PM

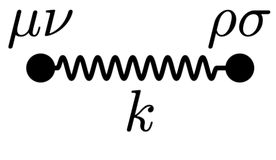
$$\Delta p_i^{(5)\mu} = m_1 m_2 \left(m_1^4 \Delta p_{0SF}^{(5)\mu} + m_1^3 m_2 \Delta p_{1SF}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{2SF}^{(5)\mu} + m_1 m_2^3 \Delta \bar{p}_{1SF}^{(5)\mu} + m_2^4 \Delta \bar{p}_{0SF}^{(5)\mu} \right)$$

Separating dissipative vs. conservative effects:



$$= \frac{iP_{\mu\nu;\rho\sigma}}{k^2 + sgn(k^0)i0}$$

$$\Delta p_1 = \Delta p_{diss} + \Delta p_{cons}$$



$$= \frac{iP_{\mu\nu;\rho\sigma}}{k^2 + i0}$$

Physical scattering:

➡**Retarded propagators** and causal flow

➡Full **physical answer**!

$$\Delta p_1 + \Delta p_2 = P_{rad}$$

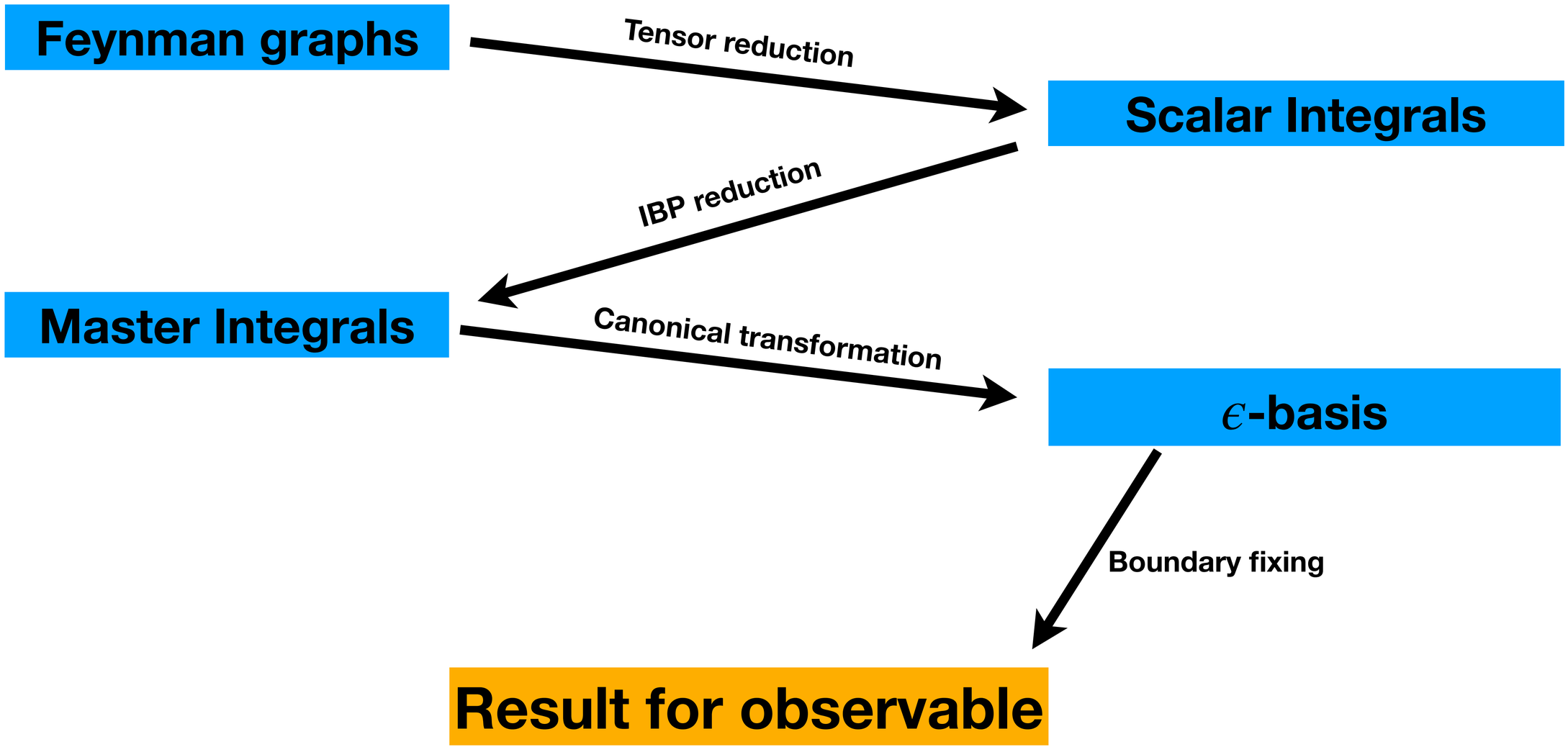
Conservative scattering (no radiation):

➡Setting boundary conditions

➡**Feynman propagators** for gravitons:

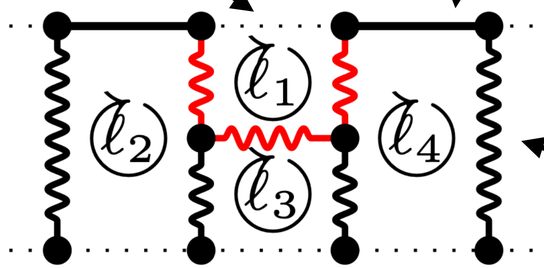
$$\Delta p_1 + \Delta p_2 = 0$$

Workflow


Feynman graphs
Tensor reduction
Scalar Integrals
IBP reduction
Master Integrals
Canonical transformation
ϵ-basis
Boundary fixing
Result for observable

Integral family at first self force order

$$\mathcal{J}^{\{\sigma\}}_{\{n\}} = \int_{l_1 \cdots \ell_4} \frac{\delta^{(\bar{n}_1-1)}(\ell_1 \cdot v_1)\delta^{(\bar{n}_1-1)}(\ell_2 \cdot v_2)\delta^{(\bar{n}_1-1)}(\ell_3 \cdot v_2)\delta^{(\bar{n}_1-1)}(\ell_4 \cdot v_2)}{\prod_{i=1}^{4} D_i^{n_i}(\sigma_i) \prod_{I<J} D_{IJ}^{n_{IJ}}}$$



Linear propagators:

$$D_i = \ell_i \cdot v_{\bar{j}_i} + \sigma_i i0^+$$

Graviton propagators:

$$D_{IJ} = (\ell_I - \ell_J)^2$$

$$D_{qi} = (\ell_i + q)^2$$

$$D_{0i} = \ell_i^2$$

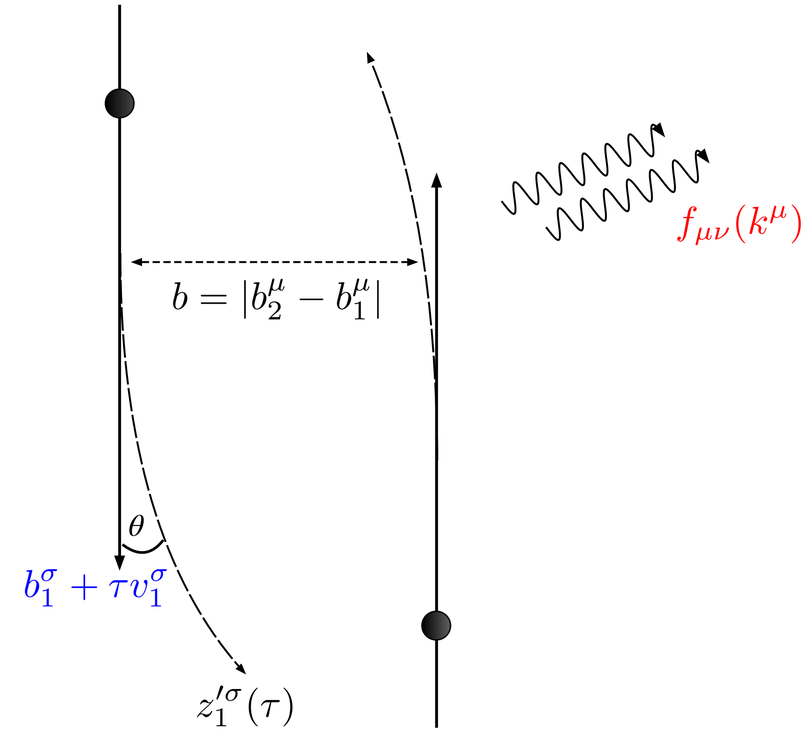
- **Single scale:** $v_1 \cdot v_2 = \gamma = 1/2(x + x^{-1})$
- Retarded propagator make integrals (pseudo-)real

“Parity” of integrals

Observables:

θ $\qquad$ E_{rad}

$$\Delta p^\mu = c_b b^\mu + c_{v_1} v_1^\mu + c_{v_2} v_2^\mu$$

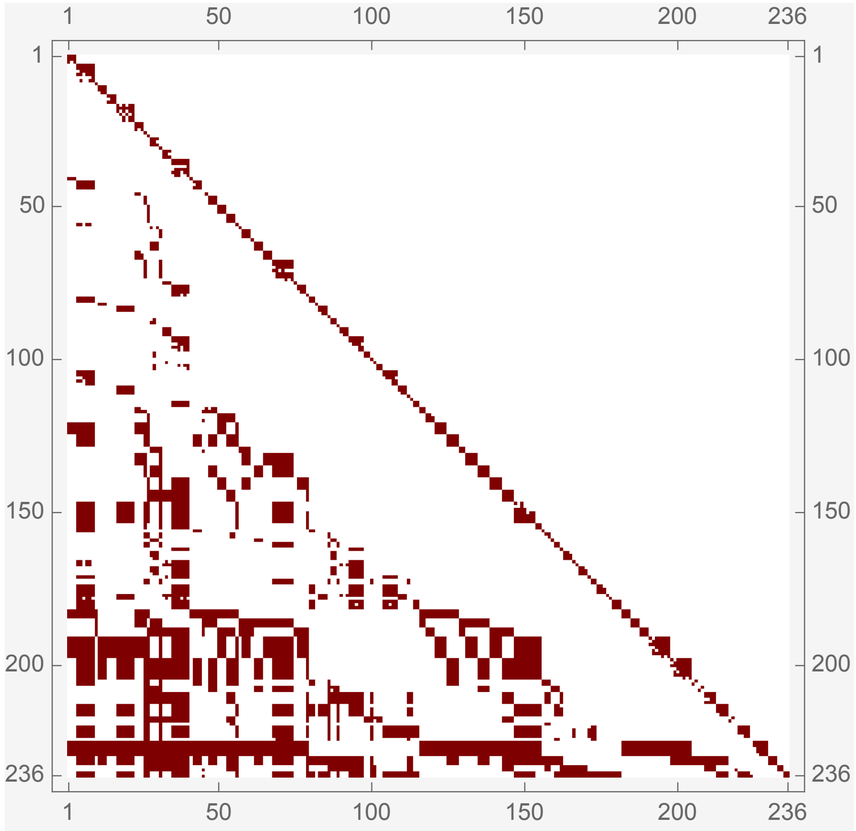


- Scattering angle θ
- Integrals purely **real**, even number of lin. prop.
- Conservative + dissipative effects

- Radiated energy E_{rad}
- Integrals purely **imaginary**, odd number of lin. prop.
- Dissipative effects + lower order results

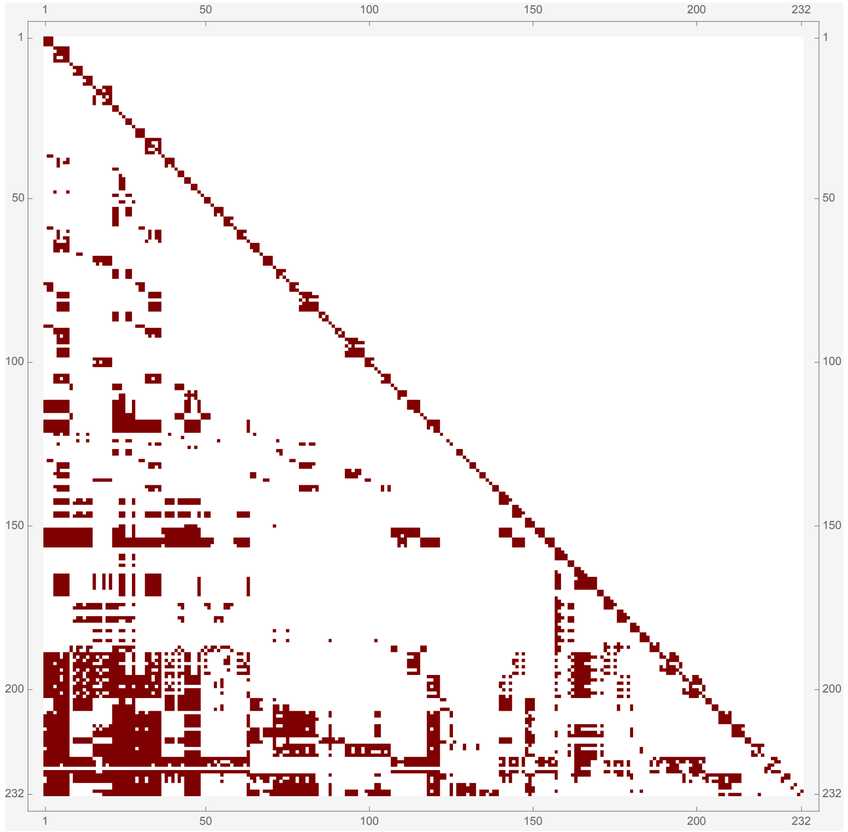
Two distinct integral families of 232+234 masters with same propagators

Differential Equations

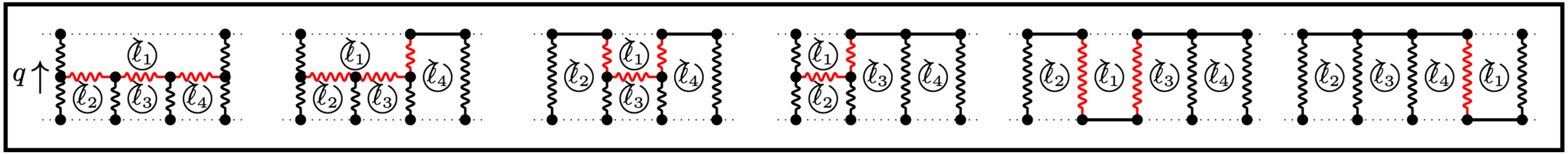


Even parity

$$\frac{d\vec{I}}{dx} = M(x,\epsilon)\vec{I}$$

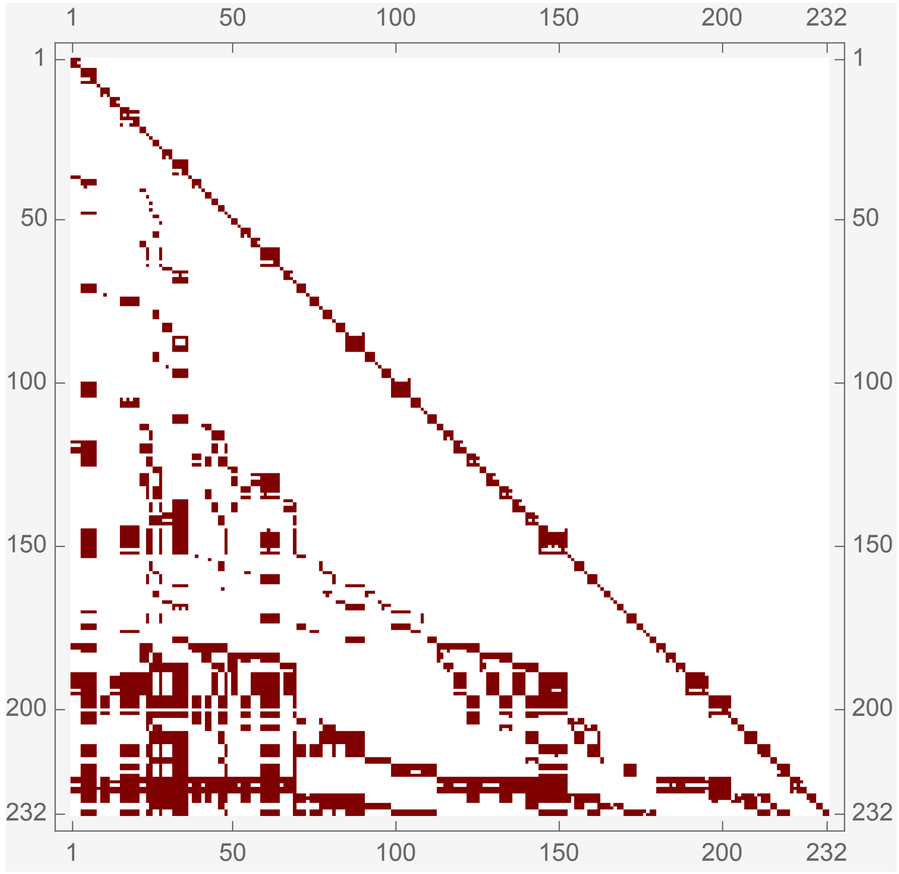


Odd parity

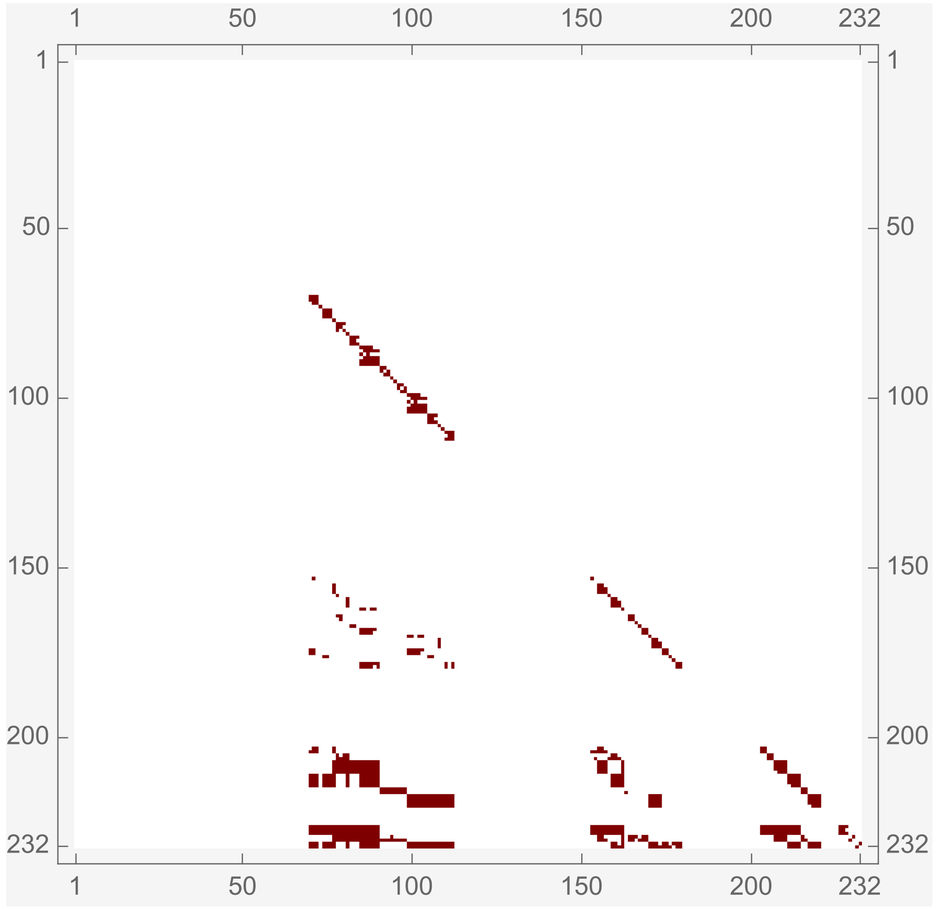


Differential Equations

Integrals contributing to **radiated energy**



Retarded propagator



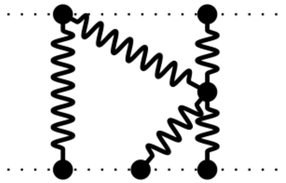
Feynman propagator

CY in the Sky

$$\theta = z\frac{\mathrm{d}}{\mathrm{d}z}, z = x^2$$

4PM K3 sector:

(Dlapa,Kaelin,Liu, Porto) [2106.082776]
(Bern,Parra-Martinez, Robin, Ruf, Shen, Solon, Zeng) [2101.07254]



$$= \pi^4 \epsilon x K^2 (1 - x^2) + \mathcal{O}(\epsilon^2)$$

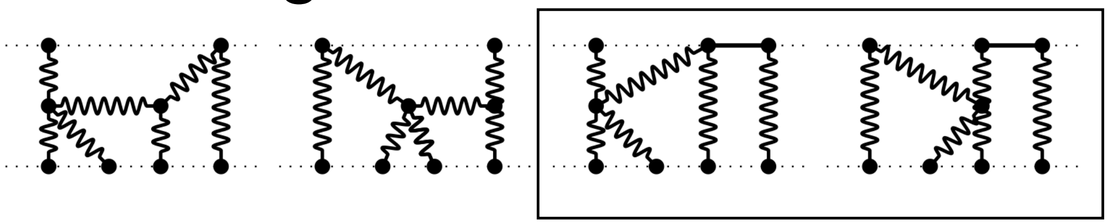
$$\mathscr{L}^{(3)} = \mathrm{Sym}^2(\mathscr{L}^{(2)})$$

$$\mathscr{L}^{(2)}_1 = \theta^2 - z(\theta + \frac{1}{2})^2$$

➡ **Solution factorizes into elliptic functions of first and second type**

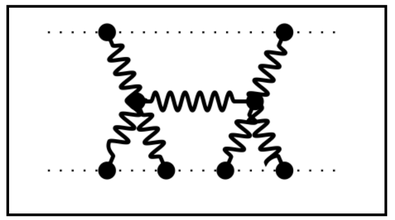
5PM-1SF K3 sectors:

Odd integrals:



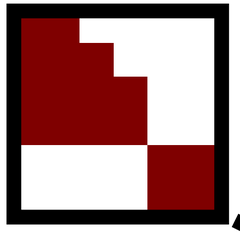
Conservative contribution

Even integrals:

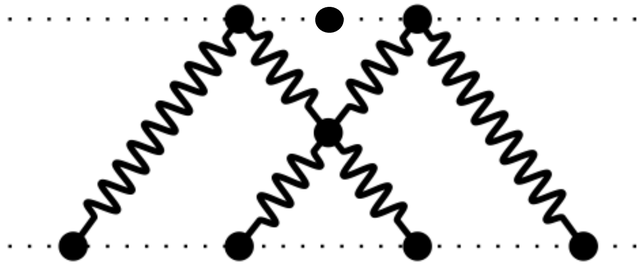


$$= -1/(1024\pi^2) x K^2 (1 - x^2) + \mathcal{O}(\epsilon)$$

CY in the Sky



CY3 geometry: (Klemm, Nega, **BS**, Plefka) [2401.07899]



$$\mathscr{L}_1^{(4)} = \theta^4 - 2^8 z(\theta + \frac{1}{2})^4$$

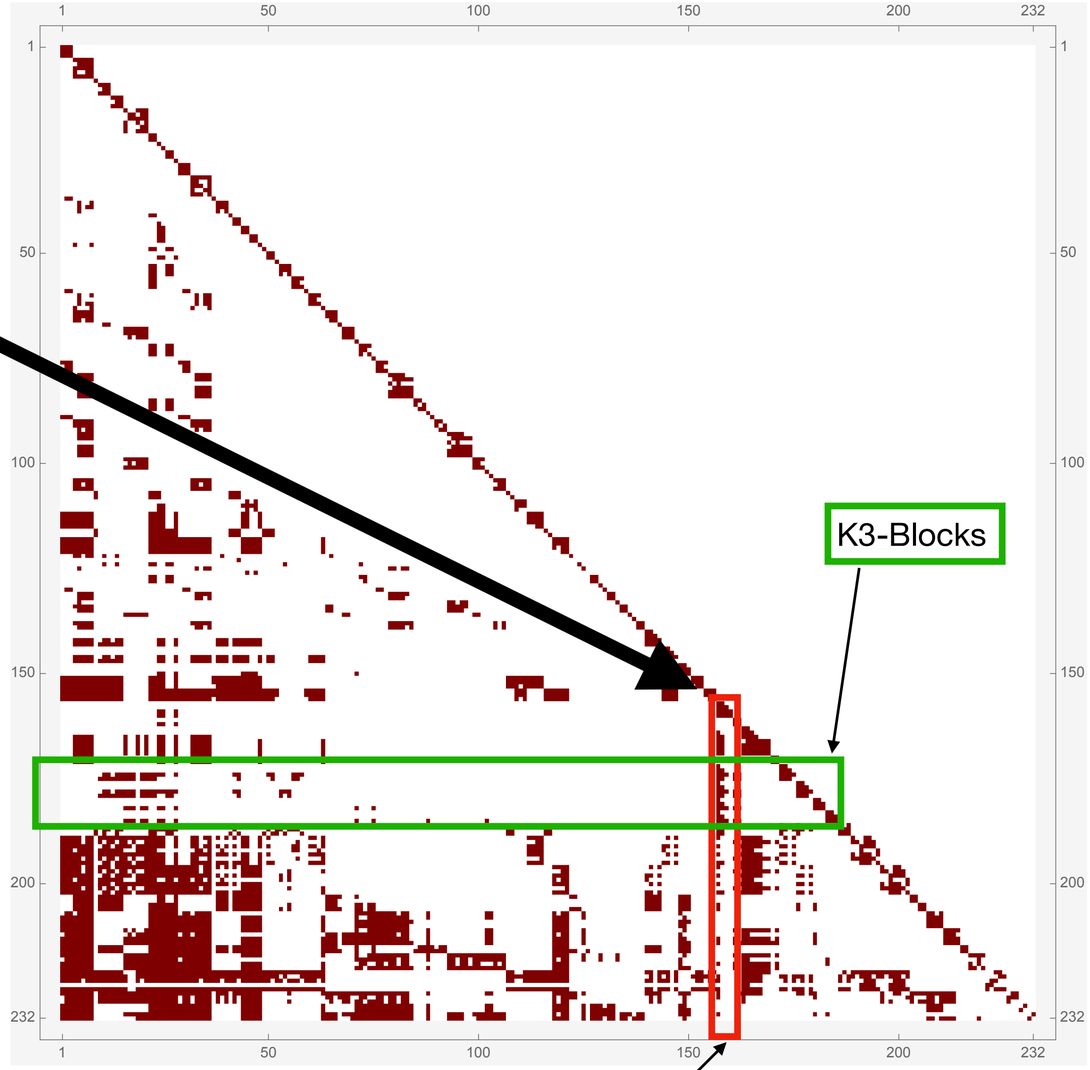
$$\mathscr{L}_1^{(4)} = \mathscr{L}_1^{(2)} * \mathscr{L}_1^{(2)}$$

➡**Vanishing for Feynman propagator!**

➡K3 and CY3 connected to same elliptic curve

➡ϵ-form by unipotent+semi-simple split (Goerges,Nega,Tancredi,Wagner [2305.14090])

➡Using IBPs+algorithms to find good integrals on off-diagonals (Dlapa,Kaelin,Liu,Porto [2304.01275])



Results

Conservative scattering angle:

Driesse, Jakobsen, Mogull, Plefka, BS, Usovitsch [2403.07781]:

$$\frac{\theta}{\Gamma} = \sum_k d_k(x) + \nu \sum_{k=1}^{31} c_k(x) f(x)$$

0SF 1SF

- **Elliptics drop out** (they were present at 4PM)
- also seen in $\mathcal{N} = 8$ SUGRA (Bern, Herrmann, Roiban, Ruf, Smirnov, Zeng) [2406.01554]

Function space:

	3PM-1SF	**4PM-1SF**	**5PM-1SF**
Kernels of iterated integrals	$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}$	$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{x}{x^2+1}$	$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{x}{x^2+1}$
Max. Weight	1	2	3
Functions from rotation		$K^2, E^2, E \cdot K$	

Results

Dissipative results (preliminary):

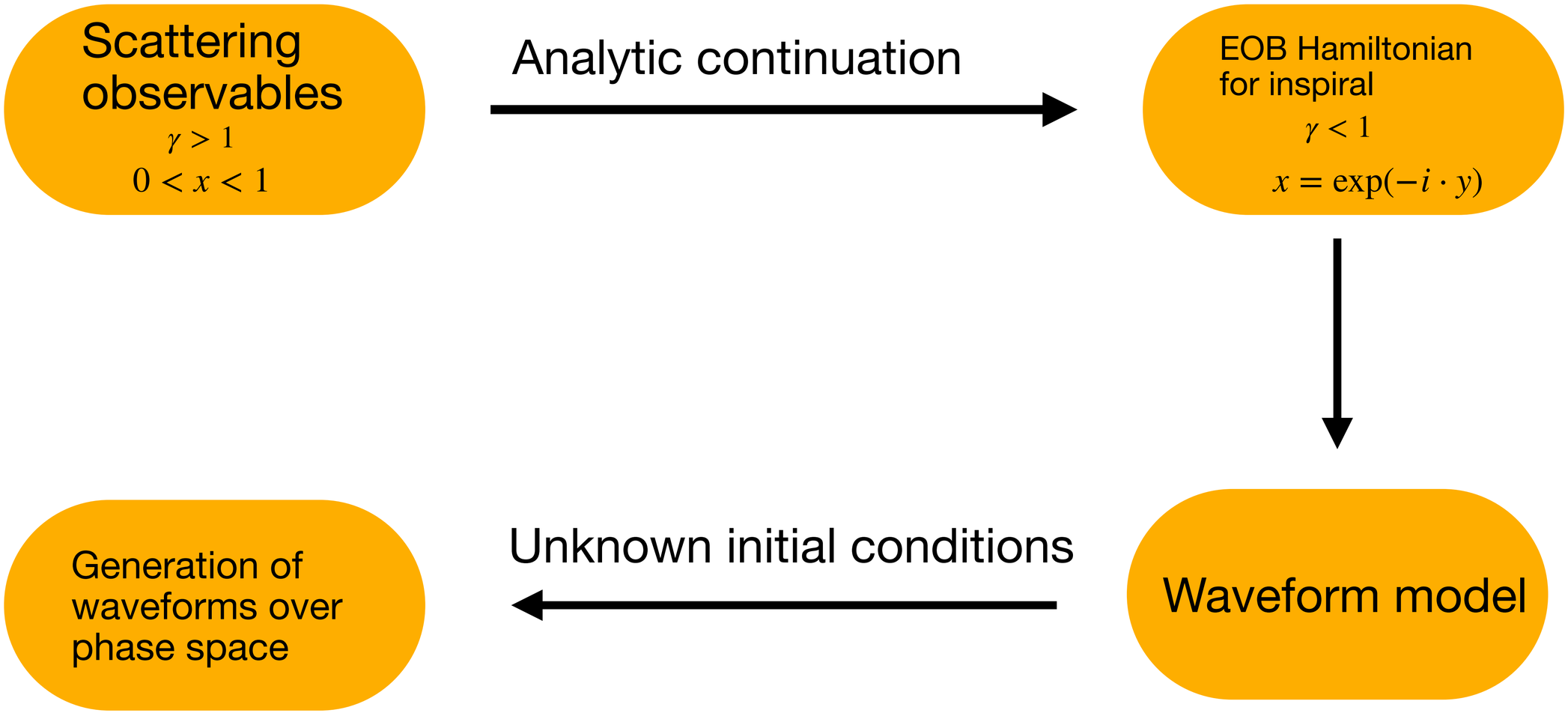
Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, BS, Usovitsch [2409.***]:**

$$\Delta p_{diss} = \nu \sum c_k(x) f(x)$$

Function space:

	3PM-1SF	4PM-1SF	5PM-1SF
Kernels of iterated integrals	$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}$	$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{x}{x^2+1}$	$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{x}{x^2+1}$ $+g(K^2, \omega_{0,CY3}, \alpha, G_i \dots)$
Max. Weight	1	2	4
Functions from rotation			$K^2, E^2, E \cdot K, \omega_{0,CY3}, \alpha, G_i$ $\omega'_{0,CY3}, \alpha', G'_i, \dots$

The problem of special functions and waveform models



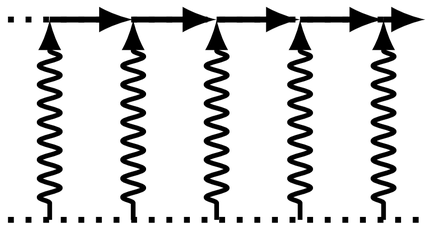
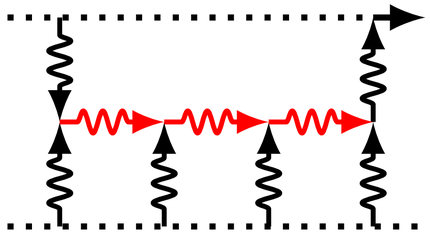
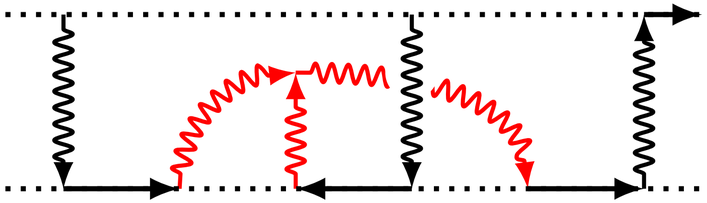
Extremely fast numerical evaluation needed!

Summary

- complete 5PM-1SF including dissipative effects
- Elliptics and beyond play an important role in classical gravitational scattering state of the art calculations
- First instance in classical gravitational scattering in which CY3 makes appearance in physical observable
- Elliptics surprisingly drop out of conservative result
- Extended special function space in dissipative sector!

Integral families at 4 loop

$$\nu = m_1 m_2/(m_1+m_2)$$

0SF - geodesic motion	1SF - $\mathcal{O}(\nu)$	2SF - $\mathcal{O}(\nu^2)$
		
• rational functions • 1 planar family • Known to all orders in G • 60 MIs	• Polylogs, K3, CY3… • 1 planar family • 45 Topsectors • ~460 MIs	• Polylogs, K3, CY3… • planar + non-planar • No results yet • >1000 MIs

$$\mathcal{J}^{\{\sigma\}}_{\{n\}} = \int_{l_1 \cdots \ell_L} \frac{\delta^{(\bar{n}_1-1)}(\ell_1 \cdot v_{j_1})\delta^{(\bar{n}_1-1)}(\ell_2 \cdot v_{j_2})\delta^{(\bar{n}_1-1)}(\ell_3 \cdot v_{j_3})\delta^{(\bar{n}_1-1)}(\ell_4 \cdot v_{j_4})}{\prod_{i=1}^{L} D_i^{n_i}(\sigma_i)\prod_{I<J} D_{IJ}^{n_{IJ}}}$$

$$D_i = \ell_i \cdot v_{\bar{j}_i} + \sigma_i i0^+$$
$$D_{IJ} = (\ell_I - \ell_J)^2$$
$$D_{qi} = (\ell_i + q)^2$$
$$D_{0i} = \ell_i^2$$

Single scale: $v_1 \cdot v_2 = \gamma$