Some aspects of Siegel modular forms one might encounter when working with scattering amplitudes

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- The basics of and perspectives on Siegel modular forms.
- Fourier and Fourier-Jacobi coefficients. (Use as cross-check?)
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- The basics of and perspectives on Siegel modular forms.
- Fourier and Fourier-Jacobi coefficients. (Use as cross-check?)
- Real-analytic Siegel modular forms.
- Missing: Computation of Siegel modular forms.

The basics of Siegel modular forms

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Here focus on role as generating series.

Upper half space

$$\begin{aligned}\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathbb{H} &:= \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\},\\ \mathrm{Sp}_g(\mathbb{R}) \circlearrowright \mathbb{H}^{(g)} &:= \{\tau \in \mathrm{Mat}_g^{\mathrm{t}}(\mathbb{C}) : \mathrm{Im}(\tau) > 0\}.\end{aligned}$$

The imaginary part often viewed as a positive definite quadratic form.

The symplectic group is the isotopy group of the symplectic form represented by

$$\omega^{(g)} = \begin{pmatrix} 0 & -\mathrm{id}_g \\ \mathrm{id}_g & 0 \end{pmatrix}, \quad \text{that is } {}^{\mathrm{t}}\gamma\omega^{(g)}\gamma = \omega^{(g)}.$$

Generalized Möbius transformations

Using block entries is convenient and makes formulas very similar to the elliptic case:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{R}), \quad {}^{\mathrm{t}}ac = {}^{\mathrm{t}}ca,\ {}^{\mathrm{t}}bd = {}^{\mathrm{t}}db,\ {}^{\mathrm{t}}ad - {}^{\mathrm{t}}cb = \mathrm{id}_g.$$

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Generalized Möbius transformations give the action on the upper half space:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau := (a\tau + b)(c\tau + d)^{-1}.$$

Homogeneous space

The Siegel upper half space is a homogeneous space.

The stabilizer of $i\mathrm{id}_g \in \mathbb{H}^{(g)}$ is

$$\mathrm{K} := \left\{ \left(\begin{smallmatrix} d & -c \\ c & d \end{smallmatrix}\right) : ci + d \in \mathrm{U}_g(\mathbb{R}) \right\} \cong \mathrm{U}_g(\mathbb{R}).$$

Notation

We write functions as $f(\tau)$, even if they are not holomorphic. In physics, this might be $f(\tau, \overline{\tau})$.

We write $x = \mathrm{Re}(\tau)$, $y = \mathrm{Im}(\tau)$.

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- polynomials in the entries of y with holomorphic functions as coefficients,
- solutions to differential equations studied by Shimura,
- solutions to differential equations studied in the 90ies by the school of Oda (large discrete series).

Slash actions

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In the literature on automorphic representation theory only one component of f is usually studied, but these notions are interchangeable.

Example: Rational representations

For $\mathrm{GL}_g(\mathbb{C})$ the subrepresentations of $\det^k \mathrm{std}^l$, "l tensors", for $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$ are the rational ones (but the language extends to all algebraic groups). For instance, a common realization of sym^2 on $\mathrm{Mat}_g^{\mathrm{t}}(\mathbb{C})$ is

$$\mathrm{sym}^2(c\tau + d)\, m = {}^{\mathrm{t}}(c\tau + d)\, m (c\tau + d).$$

Definition

A holomorphic function $f : \mathbb{H}^{(g)} \longrightarrow V(\sigma)$ is a Siegel modular form of weight σ and genus $g \geq 2$ if

$$f|_\sigma \gamma = f \quad \text{for all } \gamma \in \mathrm{Sp}_g(\mathbb{Z}).$$

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Other groups like the paramodular groups are possible, and they are genuinely important.

Relation to curves

Siegel modular forms are sections of vector bundles over $\mathscr{A}_g$.

The classical Torelli map gives

$$\mathscr{M}_g(\mathbb{C}) \longrightarrow \mathscr{A}_g(\mathbb{C}),$$

surjective for $g \in \{2,3\}$ and co-dimension one if $g = 4$.

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As stacks this is 2-to-1 outside of the hyperelliptic locus (see Stacks Project, Lemma 109.16.5 (1)(b)).

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- A definition of harmonic weak Siegel Maaß forms.
- Some generating series incl. one for Gromov-Witten invariants of $K3 \times E$.
- Real-analytic integrals like the Kawazumi Zhang invariant.
- Katz-Moody-Borcherds denominator formulas (rather orthogonal modular forms).

Example: Eisenstein series

Eisenstein series are associated with a Siegel modular form f of genus $0 \leq h < g$, where forms of genus 0 are constants.

$$\sum_{\gamma \in \Gamma_{\underline{\infty}(h)} \backslash \mathrm{Sp}_g(\mathbb{Z})} f|_k \gamma$$

for the Klingen parabolic subgroups

$$\Gamma_{\underline{\infty}(h)} = \left\{ \begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & \mathrm{id}_h & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & \mathrm{id}_h \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z}) \right\}, \quad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \mathrm{Sp}_{g-h}(\mathbb{Z}).$$

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Construction in literature only for scalar weights.

Example: Theta series

Theta series are generating series of sublattices in an Euclidean lattice L with associated quadratic form q_L:

$$\theta_L^{(g)}(\tau) = \theta_L(\tau) := \sum_{l \in L^g} e(q_L(l)\tau), \quad e(\tau) = \exp\big(2\pi i \,\mathrm{trace}(x)\big).$$

Generalized Siegel modular forms

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Orthogonal modular forms can be viewed as generalization of the case $g = 2$, but sometimes this is misleading.

Fourier and Fourier-Jacobi coefficients

Translations in (para)modular groups

The modular groups contains translations

$$\left(f\big|_\sigma \begin{pmatrix} \mathrm{id}_g & b \\ 0 & \mathrm{id}_g \end{pmatrix}\right)(\tau) = f(\tau + b), \quad b \in \mathrm{Mat}_g^{\mathrm{t}}(\mathbb{Z}).$$

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Resulting Fourier expansion

$$f(\tau) = \sum_{t \in \mathrm{Mat}_g^{\mathrm{t}}(\mathbb{Z})^\vee} c(t) e(t\tau), \quad e(\tau) = \exp\big(2\pi i \,\mathrm{trace}(x)\big),$$

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For paramodular groups contain even further translations, resulting in extra restrictions on t (i.e. a sparser Fourier expansion).

Covariance of Fourier coefficients

Fourier coefficients exhibit extra symmetry

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In the paramodular case, u is more restricted, but there is finite "character decomposition" with respect to a 2-elementary group that gives further symmetries (only written up in genus 2 by Gritsenko; orthogonal setting).

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- a diagonal t with prime entries such that $c(t) \neq 0$ (wip with Sebastián Herrero and Yves Martin).

Cusp forms

From the Fourier coefficients we get several notions of cusp forms.
A form is an h-cusp form for $0 \leq h < g$ if

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Also referred to as f has co-rank h (Weissauer).

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Cusp forms are $(g-1)$-cusp forms in most of the literature, but sometimes 0-cusp forms.

Example: Cusp forms

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In genus at least 4, the difference

$$\theta_{D_{16}} - \theta_{E_8 \oplus E_8}$$

is a 3-cusp form (and vanishes up to genus 3). This is connected to Milner's example of isospectral tori.

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Conjecture for cusp forms

$$c(t) \ll \det(t)^{\frac{k-(g+1)}{2}+\epsilon}.$$

Converse growth bounds

One can ask whether cusp forms are characterized by the growth of their Fourier coefficients. For $k \geq 2g$ Böcherer–Das proved:

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Kitaoka for Eisenstein series associated with genus h cusp forms, $1 \leq h < g$, of weight $k \geq g + h + 1$ proved

$$c(t) \ll \det(t)^{k-\frac{g+1}{2}} / \det(t_h)^{k-\frac{h+1}{2}}$$

for t in a fixed Siegel domain and the principal minor t_h of size $h \times h$.

Fourier-Jacobi expansion

Instead of using the whole group of translations

$$\left(f\big|_\sigma \begin{pmatrix} \mathrm{id}_g & b \\ 0 & \mathrm{id}_g \end{pmatrix}\right)(\tau) = f(\tau + b), \quad b \in \mathrm{Mat}_g^{\mathrm{t}}(\mathbb{Z}),$$

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Resulting expansion is called Fourier-Jacobi expansion

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This is a reduction mechanism, since Siegel–Jacobi forms can be related to Siegel modular forms (wip with Jan Feldmann for vector-valued forms; special case $g = 1$ by Ibukiyama–Kyomura).

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and its first Fourier-Jacobi coefficients is commonly known as $\phi_{10,1}$ with

$$\phi_{10,1}(\tau, 0) = 0, \quad \partial_z^2 \phi_{10,1}(\tau, 0) = \Delta(\tau),$$

the discriminant form in genus 1.

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with the Fourier expansion symmetry

$$c(t) = \sigma({}^{\mathrm{t}}u^{-1})\, c({}^{\mathrm{t}}utu), \quad u \in \mathrm{GL}_g(\mathbb{Z})$$

converges automatically (joint with Jan Bruinier).

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Extended to other modular groups in genus ≤ 4 (also joint with Jan Bruinier).

Analytic aspects

Maaß operators

There are covariant order one differential operators R_σ, L_σ,

$$\mathrm{R}_\sigma f : \mathbb{H}^{(g)} \longrightarrow \mathrm{sym}^2 \otimes V(\sigma),$$
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Essentially due to Maaß, different formulation in joint work with Albrecht Klemm, Maximilian Poretschkin, and Thorsten Schimannek.

Example: Maaß operators

In genus 1 and weight k, we have

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In higher genus:

$$\mathrm{L}_\sigma = y^{\mathrm{t}}\big(y\partial_{\overline{\tau}}\big) \quad \text{and} \quad \mathrm{R}_\sigma = \partial_\tau - \tfrac{i}{4}\sqrt{y}^{-1}(\partial\sigma + {}^{\mathrm{t}}\partial\sigma)(\mathrm{id}_g)\sqrt{y}^{-1}.$$

Harish–Chandra modules

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Annihilation by Maaß operators can be (partially) visualized by K-type support.

Harish–Chandra modules

Maaß operators change the weight, so important to track them. In automorphic representation theory built into the representations (called $(\mathfrak{g}, \mathrm{K})$-modules).

Weights correspond to representations of $\mathrm{U}_g(\mathbb{R}) \cong \mathrm{K} \subset \mathrm{Sp}_g(\mathbb{R})$. We track the irreducible ones: K-types.

Annihilation by Maaß operators can be (partially) visualized by K-type support.

From physics I have only seen modular forms coming from the Siegel induced principal series, where this picture is complete as K-types have multiplicity at most one.

Example: Harish–Chandra modules in genus 1

In genus 1 K-types form a line.

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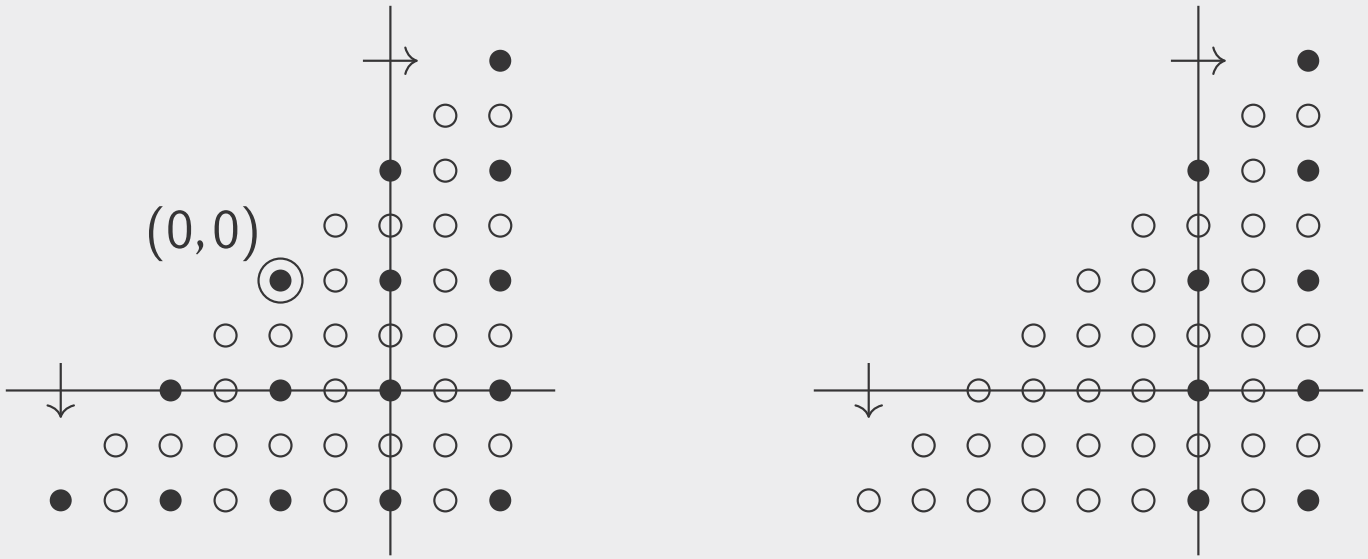
Corresponding real-analytic one in weight 0:

Example: Harish–Chandra modules in genus 2

From work on harmonic Siegel Maaß forms (should be similar for Kawazumi–Zhang invariant, the only real-analytic Siegel modular form in string theory that I know of).



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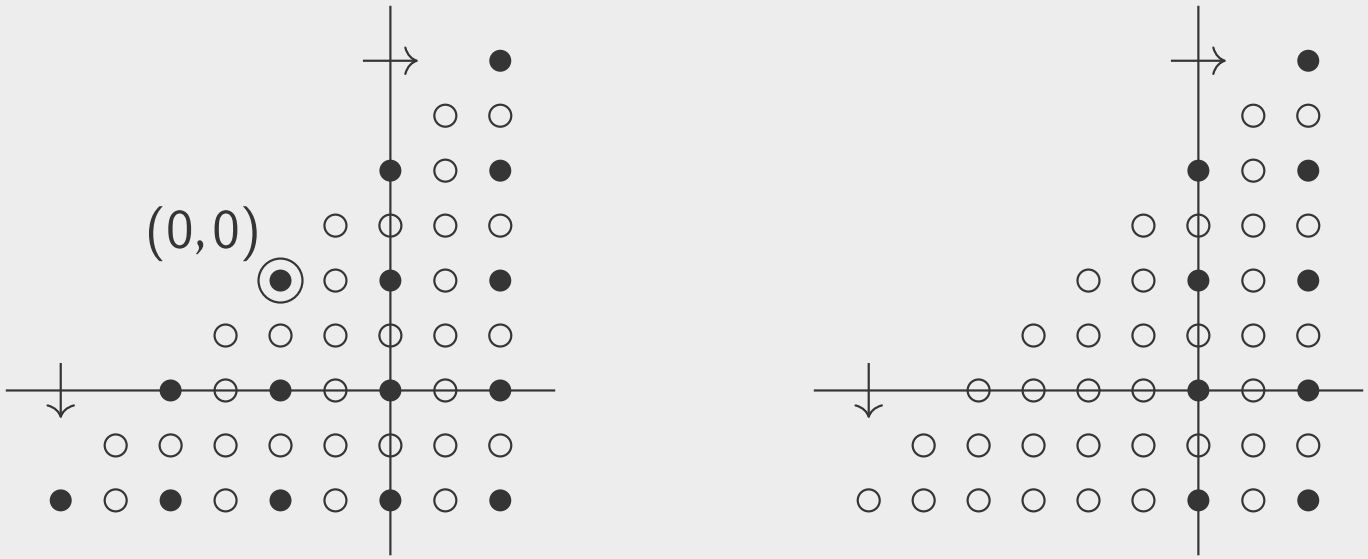
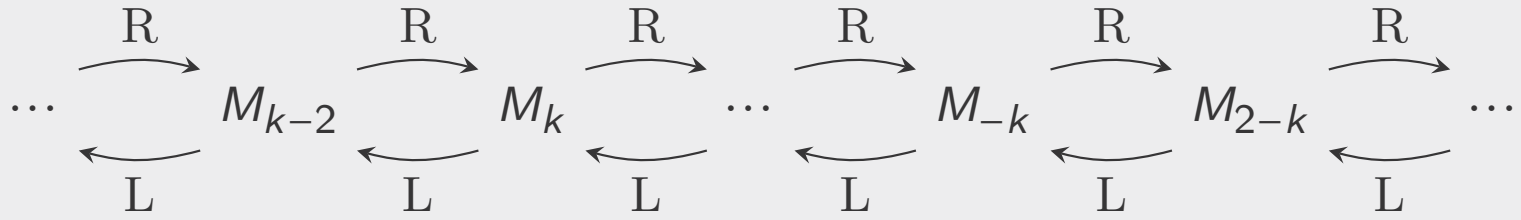


Lowest K-type in holomorphic and meromorphic cases is at the bottom left of the support.

Beyond multiplicity 1 in genus 1

In genus 1 we have a classification of all extension classes (joint with Claudia Alfes, Igor Burban). Points now represent vector spaces that form a graded module $\bigoplus M_l$:



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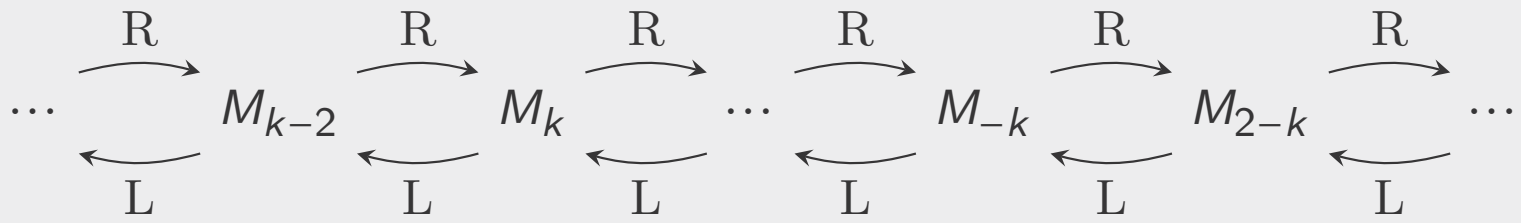
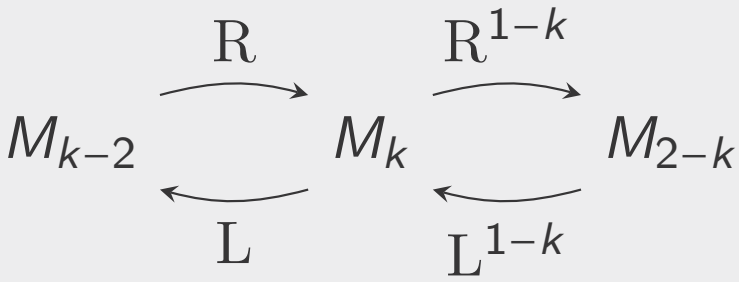


All important information is contained in a quiver representation:



Fourier-Jacobi coefficients

In the real-analytic case, the Fourier-Jacobi expansion a priori is

$$f(\tau) = \sum_{m \in \mathrm{Mat}_h^{\mathrm{t}}(\mathbb{Z})^\vee} \phi_m(\tau_1, z, y_2) e(m x_2), \quad \tau = \begin{pmatrix} \tau_1 & {}^{\mathrm{t}}z \\ z & \tau_2 \end{pmatrix}.$$

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Generalization to larger class of real-analytic Siegel modular forms stale on problem with interchanging a certain series and limit which should give $\zeta(0)$ (joint work with Olav Richter).