Motives for Feynman integrals in momentum space

Ulysse Mounoud

Université de Montpellier (IMAG)

Introduction

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- Different ways to write I_G (position, momenta, Schwinger parameters) lead to different motives.

  Schwinger parameters are already well-studied, I will focus on momenta.

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Pairing differential forms with homology classes yields the *period matrix* :

$$\left(\int_{\sigma_i} \omega_j \right)_{1 \leq i,j \leq r}.$$

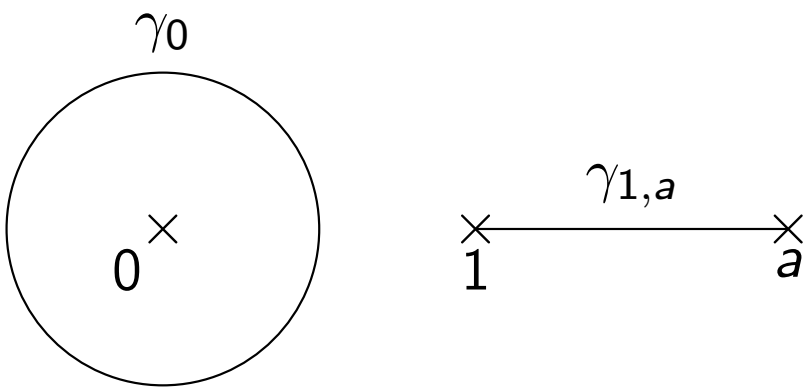
Examples

- The Lefschetz motive $\mathbb{Q}(-1) = H^1(\mathbb{C}^*)$ has period matrix

$$\begin{array}{cc} & \frac{dz}{z} \\ \gamma_0 & (2\pi i) \end{array}$$

- The Kummer motive $\mathcal{K}_a = H^1(\mathbb{C}^*, \{1, a\})$ has period matrix

$$\begin{array}{cc} \frac{dz}{a-1} & \frac{dz}{z} \\ \gamma_{1,a} & \begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix} \\ \gamma_0 & \end{array}$$



Upshot

The weight filtration on motives yields a weight filtration on periods.

Example

$\mathbb{Q}(-1)$ is of weight 2 while $\mathcal{K}_a$ mixes weight 0 and 2 :

$$0 \to \mathbb{Q}(0) \to \mathcal{K}_a \to \mathbb{Q}(-1) \to 0.$$

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There is a motivic coaction on periods :

$$\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}^{\mathfrak{dr}}.$$

Schwinger parameters

Bloch, Esnault, Kreimer [1] first, then Brown [2] have defined motives for Feynman integrals along Schwinger parameters.

Proposition (Brown [2])

Feynman integrals with bounded weight and number of kinematics generate a finite dimensional vector space.

Conjecture (Panzer, Schnetz, [3])

The ϕ_4 Feynman integrals are stable under the motivic coaction.

Proposition (Tapuskovic [4])

Ansatz for I_G and explicit formulas for the motivic coaction for 1-loop graphs and the sunrise graph.

Momentum space

We are interested in

$$I_G = \pi^{\frac{-\ell D}{2}} \int_{\mathbb{R}^{\ell D}} \frac{dk_1 \dots dk_\ell}{\prod_{j=1}^{N} (p_j(k)^2 + m_j^2)}.$$

The motive M_G is given (up to some blow-ups) by

$$H^{\ell D}\left(\mathbb{P}^{\ell D} \setminus \bigcup_{j=1}^{N} Q_j, H_\infty\right)\left(\frac{\ell D}{2}\right)$$

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Proposition

If the integral converges, I_G is a period of M_G. In particular I_G is of weight $\leq \ell D$.

Choice of compactification

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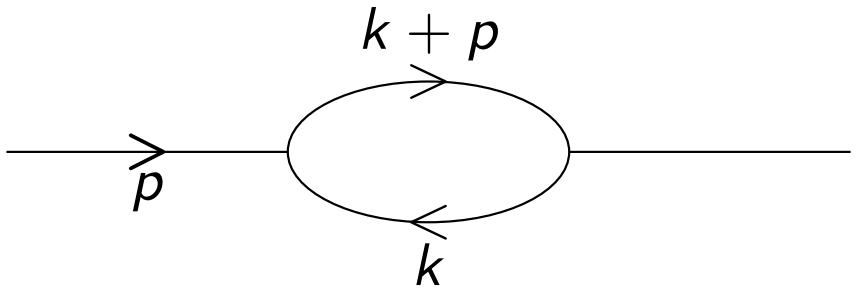
  so X is a (smooth) quadric hypersurface (see Hwa, Teplitz [5]).
- For higher loop orders one can take $m = D + \binom{\ell+1}{2}$ and

$$(k_1, \ldots, k_\ell) \mapsto [k_1 : \cdots : k_\ell : 1 : k_1^2 : k_1 k_2 : \cdots : k_\ell^2].$$

Bubble graph



$$I_G = \int_{\mathbb{R}^2} \frac{d^2k}{\pi(k^2+m^2)((k+p)^2+m^2)} \; ; M_G = H^2\left(X \setminus (Q_1 \cup Q_2)\right)(1).$$

Deligne's spectral sequence yields

$$0 \to \mathbb{Q}(0) \to M_G \to \mathbb{Q}(-1) \to 0$$

hence

$$M_G = \mathcal{K}_{f(p,m)} \text{ and } I_G = g(p,m)\log(f(p,m)).$$

One-loop computation

Proposition

Assume generic kinematics, non-vanishing masses, $\ell = 1$, D *even. Then* I_G *is a period of the motive*

- $H^D(X \setminus \bigcup Q_i)(\frac{D}{2})$ *if* $N \geq D$;
- $H^D(\widetilde{X} \setminus \left(\bigcup Q_i \cup \widetilde{X}_\infty\right), E)(\frac{D}{2})$ *if* $N > \frac{D}{2}$ *where* $\widetilde{X}$ *is the blow-up of* X *at one point,* $\widetilde{X}_\infty$ *is the strict transform of* X_∞ *and* E *is the exceptional divisor.*

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Corollary (Nickel ; Bloch, Kreimer [6])

For $D = 4$ and $\ell = 1$ we can express I_G using only dilogarithms, squares of logarithms and algebraic functions of the kinematics.

Conclusions and outlook

Conclusions :

- Definition of a motive for momentum space and weight bound;
- Better compactification for 1 loop.

Outlook :

- Schwinger trick with parameters for motivic periods.
- How does the second compactification behave for higher loop orders ? (smoothness, definition of a motive...)
- Make sense of cuts with motives (residue maps), and study the motivic coaction.
- Dimensional regularisation ?

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Appendix : Blow-ups

Blowing up X along a closed subspace Z is the universal way of making Z into a divisor E of X.

Example

The blow-up of $\mathbb{C}^2$ at the origin replaces the point with a copy of $\mathbb{P}^1$.
We get two sets of coordinates $(x, v = \frac{y}{x})$ and $(u = \frac{x}{y}, y)$.

More generally we get simple equations for blow-ups of linear subspaces of an affine space.