One-loop four-graviton in type II string

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Work in progress with Emiel Claasen

$$\int \frac{d^2\tau}{(\mathrm{Im}\tau)^5} \int \prod_{i=1}^{3} d^2\nu_i \prod_{1\le i<j\le 4} \{\exp[\frac{-\pi(\mathrm{Im}\nu_{ji})^2}{\mathrm{Im}\tau}] |\ \vartheta_1(\nu_{ji}|\tau)|\}^{\frac{1}{2}k_i\cdot k_j}.$$

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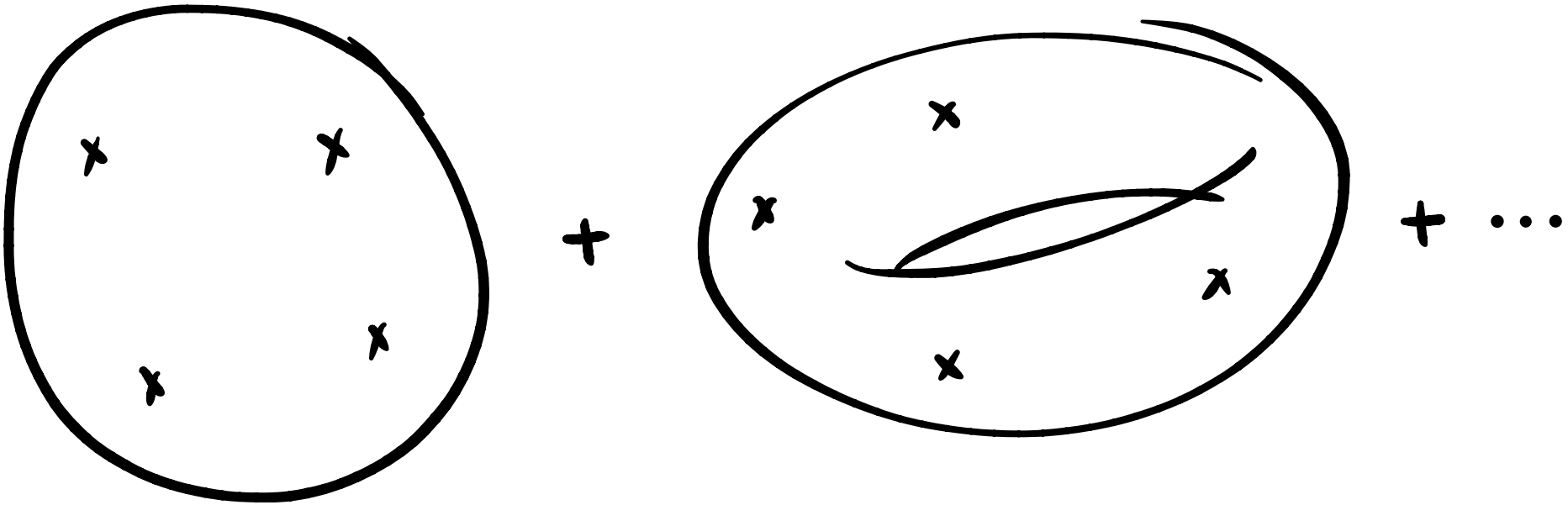
Supersymmetrical string theories - Green, Schwarz '81

Genus Expansion



$$\boldsymbol{A}(k_i, \epsilon_i) = \kappa_{10}^2 \mathcal{R}^4 \sum_{h=0}^{\infty} g_s^{2h-2} \boldsymbol{A}^{(h)}(s_{ij}),$$

Genus Zero

$$\boldsymbol{A}^{(0)}(s_{ij}) = \frac{3}{\sigma_3} \exp\left\{ 2 \sum_{m=1}^{\infty} \frac{\zeta(2m+1)}{2m+1} \sigma_{2m+1} \right\}$$

$$s_{ij} = -\alpha'(k_i + k_j)^2/4$$

$$\sigma_k = s^k + t^k + u^k$$

$$\zeta^{\mathrm{sv}}(m) = \begin{cases} 2\zeta(m), & m \quad \text{odd} \\ 0, & m \quad \text{even} \end{cases}$$

Genus One

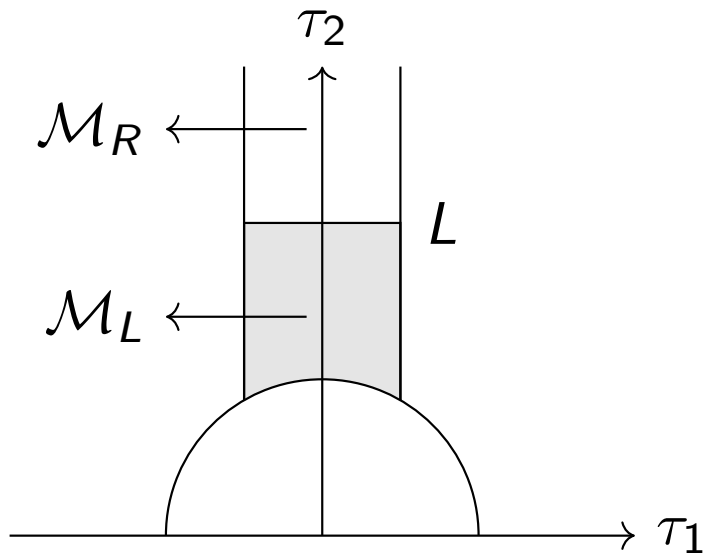
$$\boldsymbol{A}^{(1)}(s_{ij}) = 2\pi \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \mathcal{B}(s_{ij}|\tau)$$

$$\mathcal{B}(s_{ij}|\tau) = \prod_{k=2}^{4} \int_{\Sigma_\tau} \frac{d^2 z_k}{\tau_2} \exp\left\{ \sum_{1\le i<j\le 4} s_{ij} G_{ij}(\tau) \right\}$$

$$G(z,\tau) = \frac{\tau_2}{\pi} \sum_{p}{}' \frac{e^{2\pi i \langle p,z\rangle}}{|p|^2}$$

Green, Vanhove '99

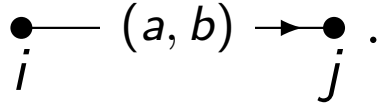
Green, Russo, Vanhove '08



$$\boldsymbol{A}^{(1)}(s_{ij}) = \mathcal{A}_{\text{an}}(s_{ij}) + \mathcal{A}_{\text{non-an}}(s_{ij})$$

Modular Graph Forms

Feynman rules:

- $\frac{\tau_2^a}{\pi^{(a+b)/2}} \sum'_p \frac{1}{p^a \bar{p}^b} \quad \longleftrightarrow \quad$ 

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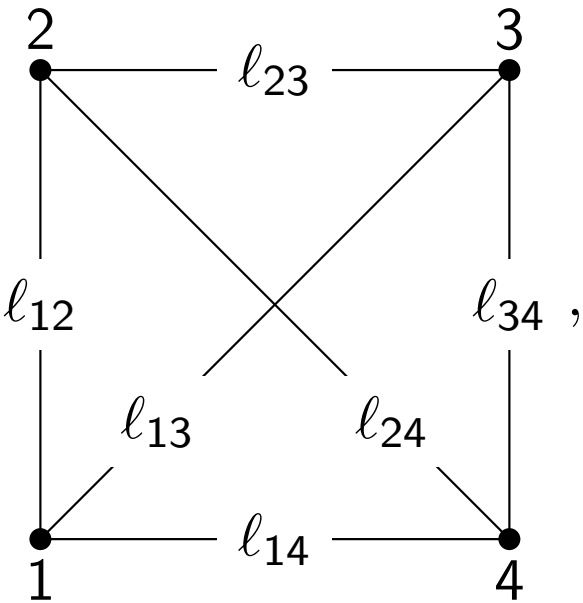
- Momentum conservation at vertices

$D_{\underline{\ell}} =$ 

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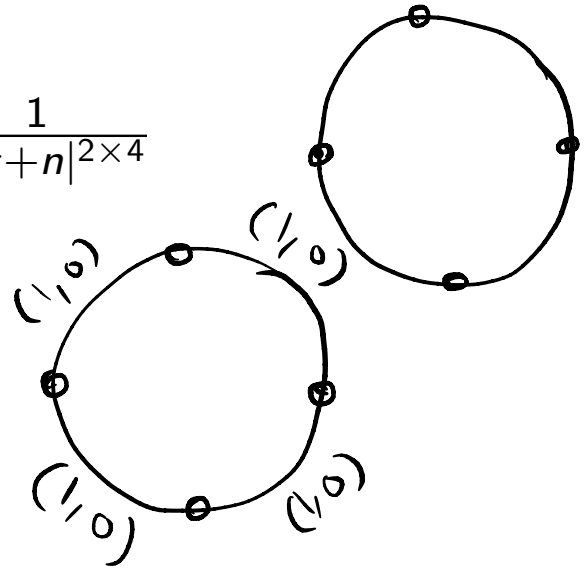
Green, Russo, Vanhove, D'Hoker, Zagier, Zerbini, Gürdogăn, Brown, Broedel, Kaidi, Duke, Gerken, Kleinschmidt, Schlotterer, Dorigoni, Hidding, Verbeek, Matthes ...

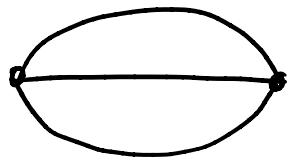
Examples

$$E_4 = \left(\frac{\tau_2}{\pi}\right)^4 \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{|m\tau+n|^{2\times 4}}$$

$$G_4 = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\tau+n)^4}$$



$$C_{1,1,1} = \left(\frac{\tau_2}{\pi}\right)^3 \sum{}'_{m_1,n_1,m_2,n_2} \frac{1}{|m_1+n_1\tau|^2|m_2+n_2\tau|^2|m_1+m_2+(n_1+n_2)\tau|^2}$$

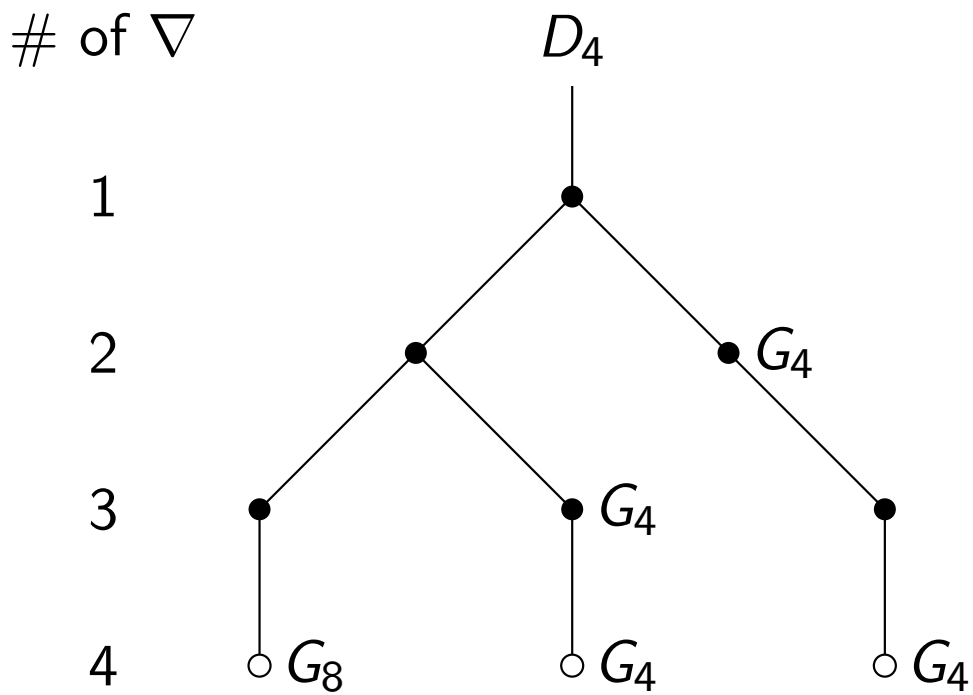


$$
\begin{aligned}
\mathcal{B}(s_{ij}|\tau) =& 1 + \sigma_2 E_2 + \sigma_3 \left(\tfrac{1}{3} D_3 + \tfrac{4}{3} E_3\right) \\
&+ \sigma_2^2 \left(\tfrac{1}{24} D_4 + \tfrac{1}{4} E_4 + \tfrac{3}{8} E_2^2\right) + \sigma_2 \sigma_3 \\
&\times \left(\tfrac{1}{72} D_5 + \tfrac{2}{9} D_{3,1,1} - \tfrac{1}{6} D_{2,2,1} + \tfrac{1}{6} D_{2,1,1,1}\right. \\
&\left. - \tfrac{1}{3} D_{1,1,1,1,1} + \tfrac{7}{36} D_3 E_2 + \tfrac{2}{3} E_2 E_3\right) + \mathcal{O}(\alpha'^6)
\end{aligned}
$$

Idea: Sieve Algorithm

D'Hoker, Green '16
Claasen, MD in progress

$$\nabla \quad = \quad 2i(\mathrm{Im}\tau)^2\partial_\tau$$



Iterated Integral of Eisenstein Series

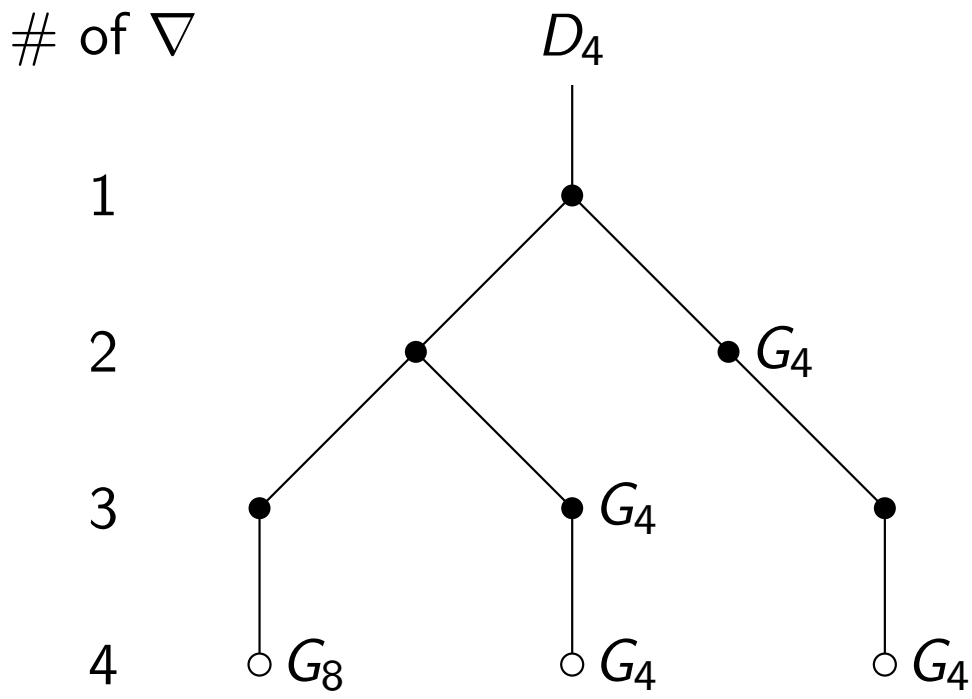
$$\pi\nabla\beta^{\mathrm{eqv}}\begin{bmatrix} j_1 & \cdots & j_\ell \\ k_1 & \cdots & k_\ell \end{bmatrix};\tau\Big]$$

$$= -\frac{1}{4}\sum_{i=1}^{\ell}(k_i - j_i - 2)\beta^{\mathrm{eqv}}\begin{bmatrix} j_1 & \cdots & j_i+1 & \cdots & j_\ell \\ k_1 & \cdots & k_i & \cdots & k_\ell \end{bmatrix};\tau\Big]$$

$$+ \frac{1}{4}\delta_{j_\ell,k_\ell-2}(2i\tau_2)^{k_\ell}G_{k_\ell}(\tau)\beta^{\mathrm{eqv}}\begin{bmatrix} j_1 & \cdots & j_{\ell-1} \\ k_1 & \cdots & k_{\ell-1} \end{bmatrix};\tau\Big] + \cdots$$

$$\beta^{\mathrm{eqv}}\begin{bmatrix} j_1 & \cdots & j_\ell \\ k_1 & \cdots & k_\ell \end{bmatrix};\frac{-1}{\tau}\Big] = \bar{\tau}^{\sum k_i - 2 - 2j_i}\beta^{\mathrm{eqv}}\begin{bmatrix} j_1 & \cdots & j_{\ell-1} \\ k_1 & \cdots & k_{\ell-1} \end{bmatrix};\tau\Big]$$

Gerken, Kleinschmidt, Schlotterer '19 '20
Dorigoni, MD, Drewitt, Hidding, Kleinschmidt, Matthes, Schlotterer, Schneps, Verbeek '22 '24



$$D_4 = -504\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}3\\8\end{smallmatrix}\right] + 216\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}1&1\\4&4\end{smallmatrix}\right] - 432\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}2&0\\4&4\end{smallmatrix}\right].$$

Change of Basis

$$(\Delta - s(s-1))F = \textit{source}$$

Dorigoni, Kleinschmidt, Schlotterer '21
DDDHKSSV '24
MD '23
Up to depth 3 $\quad\rightarrow\quad$ $(\alpha')^7$

Analytic

$$\mathcal{A}_{\text{an}}(s_{ij}) = \frac{2\pi^2}{3}\left(1 + \frac{\zeta(3)\sigma_3}{3} + \frac{29\zeta(5)\sigma_2\sigma_3}{180} + \frac{\zeta(3)^2\sigma_3^2}{18} + c\,\zeta(7)\sigma_2^2\sigma_3 + \mathcal{O}(s_{ij}^8)\right),$$

$$c = \frac{1}{4032}\left(-56d^{(0)}_{4,2,1} - 84d^{(0)}_{2,2,2,1} + 56d^{(0)}_{3,2,1,1} + 84d^{(0)}_{4,1,1,1} + 336d^{(0)}_{2,2,1,1,1} - 448d^{(0)}_{3,1,1,1,1} + 477\right),$$

Uniform transcendentality

Non-analytic

$$\mathcal{A}_{\text{non-an}}(s_{ij}) = \frac{2\pi^2}{3}\left(\hat{\mathcal{A}}_{\text{sugra}} + \hat{\mathcal{A}}_4 + \hat{\mathcal{A}}_6 + \hat{\mathcal{A}}_7 + \mathcal{O}(s_{ij}^8)\right),$$

$$\hat{\mathcal{A}}_4 = \frac{2\zeta(3)}{15}\left[\frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} - \gamma_E + \frac{63}{20}\right]\sigma_2^2 - \frac{4}{15}\zeta(3)s^4\log(-2\pi s) + 2 \text{ c.p. of } (s,t,u)$$

$$\hat{\mathcal{A}}_6 = -\frac{84s^6 + 2s^4\sigma_2}{420}\zeta(5)\log(-2\pi s) + 2\,\text{c.p. of } (s,t,u)$$
$$+ \frac{\zeta(5)}{630}\left[-35\frac{\zeta'(2)}{\zeta(2)} + 70\frac{\zeta'(4)}{\zeta(4)} - 33\frac{\zeta'(5)}{\zeta(5)} - 2\frac{\zeta'(6)}{\zeta(6)} - 33\gamma_E + \frac{16923}{140}\right]\sigma_2^3$$
$$+ \frac{\zeta(5)}{270}\left[5\frac{\zeta'(2)}{\zeta(2)} - 10\frac{\zeta'(4)}{\zeta(4)} - 18\frac{\zeta'(5)}{\zeta(5)} + 23\frac{\zeta'(6)}{\zeta(6)} - 18\gamma_E + \frac{24319}{420}\right]\sigma_3^2,$$

Non-analytic

$$\hat{\mathcal{A}}_7 = \frac{-26s^7 + s^5\sigma_2}{210}\zeta(3)^2 \log(-2\pi s) + 2\,\text{c.p. of } (s,t,u)$$
$$+ \frac{\zeta(3)^2}{630}\left[-28\gamma_E - 58\frac{\zeta'(3)}{\zeta(3)} + 28\frac{\zeta'(4)}{\zeta(4)} + 15\frac{\zeta'(6)}{\zeta(6)} + \frac{39563}{315}\right]\sigma_2^2\sigma_3.$$

Transcendentality

$$Z_k^{\mathrm{DG}} = \frac{\zeta(k)'}{\zeta(k)} - \frac{\zeta(k-1)'}{\zeta(k-1)} - \gamma_E$$

$$Z_k^{\mathrm{new}} = \frac{\zeta(k)'}{\zeta(k)} - k\gamma_E + \text{Potential Harmonic Sums } f(k)$$

What is this number?

Do we have uniform transcendentality?

$$\frac{d}{d\epsilon} \log\left(\zeta(k+\epsilon)\Gamma(k(1+\epsilon))\right)\Big|_{\epsilon=0} = \frac{\zeta(k)'}{\zeta(k)} - k\gamma_E + kH_1(k-1)$$

Thanks for listening!