Geometries in perturbative QFT

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Special Functions, MIAPbP Garching, August 8, 2024

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The period

Let G be a connected graph with

- $|E_G| = 2h_G$ and
- $|E_g| > 2h_g$ for every proper subgraph $\emptyset < g < G$ of G.

Assign a variable x_e to every edge $e \in E_G$. Let

$$\Psi_G(x) = \sum_{T \text{ span. tree}} \prod_{e \notin T} x_e$$

be the graph (Kirchhoff) polynomial of G. Then,

$$P_G = \int_{x_e > 0} \frac{1}{\Psi_G(x)^2}\Big|_{x_{|E_G|}=1} \prod_{e=1}^{|E_G|-1} \mathrm{d}x_e \in \mathbb{R}_+$$

is finite. P_G is the Feynman period of the graph G.

ϕ^4 periods

If every vertex in G has degree ≤ 4, then G is a 4-point graph with truncated external legs in four-dimensional ϕ^4 theory.
The Feynman period is a renormalization scheme independent contribution to the ϕ^4 beta-function.
Examples:

- A double line has period $P_{\bullet\!-\!\bullet} = 1$.
- The tetrahedron K_4 has $P_{K_4} = 6\zeta(3) = 6\sum_{k=1}^{\infty} k^{-3}$.

The Feynman integral of P_G has an interesting geometric structure which is reflected in the number content of P_G (D.J. Broadhurst, D. Kreimer).
With the method of graphical functions (M. Borinsky, OS) it became possible to calculate many Feynman periods. A connection between QFT and motivic Galois theory was discovered (F.C.S. Brown, E. Panzer, OS).

Feynman geometries

The calculation of periods is limited to some hyperlogarithmic cases (MZVs and their extensions). The geometry of hyperlogarithms is the punctured Riemann sphere. Feynman periods have more general geometries (F.C.S. Brown, OS). One would like to have direct access to the geometry of P_G.
Study the zero locus of the graph polynomial

$$X_G = \{\Psi_G(x) = 0\}.$$

One obtains some information about X_G by counting the number of points on X_G over finite fields $\mathbb{F}_q$,

$$[X_G]_q = |X_G(\mathbb{F}_q)|.$$

The c_2 invariant

M. Kontsevich informally conjectured that for every graph G one has $q \mapsto [X_G]_q \in \mathbb{Z}[q]$. This is true for graphs with $|E_G| \leq 12$ edges (J.R. Stembridge) but false in general (D. Doryn, OS).
In fact X_G is of general type (P. Belkale, P. Brosnan) which implies that one should see the point-count of any variety in $[X_G]_q$ for some (potentially large) graph G.
The connection to P_G is only in the lowest 'coefficient' of $q \mapsto [X_G]_q$ (D. Doryn, F.C.S. Brown, OS). More precisely,

$$[X_G]_q \equiv 0 \mod q^2$$

if G has at least three vertices. Define the c_2 invariant

$$c_2^{(q)}(G) \equiv \frac{[X_G]_q}{q^2} \mod q$$

and consider $c_2(G)$ as an infinite sequence $q \mapsto \mathbb{Z}/q\mathbb{Z}$ (OS).
Example: $c_2(K_4) \equiv -1 \mod q$ for all q.

Dodgson polynomials

The graph polynomial is linear in each variable. In particular,

$$\Psi_G(x) = \Psi^{12}x_1x_2 + \Psi^1_2x_1 + \Psi^2_1x_2 + \Psi_{12},$$

where the $\Psi^{\bullet}_{\bullet}$ depend on $x_3, \ldots, x_{|E_G|}$. We have

$$\Psi^1_2\Psi^2_1 - \Psi^{12}\Psi_{12} = (\Psi^{1,2})^2$$

for some function $\Psi^{1,2}$ which is linear in $x_3, \ldots, x_{|E_G|}$. We define

$$\Psi^{1,2}(x) = \Psi^{13,23}x_3 + \Psi^{1,2}_3$$

and obtain (F.C.S. Brown, OS)

$$c_2^{(q)}(G) \equiv -[\Psi^{13,23}\Psi^{1,2}_3]_q \mod q.$$

Denominator reduction

It is easy to see that

$$[(Ax+B)(Cx+D)]_q \equiv -[AD-BC]_q \mod q$$

for any constants (in x) A, B, C, D. Repeatedly taking resultants in 'linear' variables defines a sequence of 'denominators' ${}^n\Psi_G$ which depend on the set (not the sequence) of eliminated variables. With

$${}^3\Psi_G = \Psi^{13,23}\Psi_3^{1,2}$$

for the set $\{x_1, x_2, x_3\}$ we obtain

$$c_2^{(q)}(G) \equiv (-1)^n[{}^n\Psi_G]_q \mod q.$$

The functions ${}^n\Psi_G$ become less complex with increasing n (F.C.S. Brown, OS).

Quadratic denominator reduction

Denominator reduction is a good tool to simplify the calculation of initial sequences of $c_2(G)$.
At some stage, however, denominator reduction terminates because one obtains an irreducible polynomial ${}^n\Psi_G$ which is quadratic in all remaining variables.
One can proceed by the transition to the Legendre symbol

$$\left(\frac{A}{q}\right) = |\{x \in \mathbb{F}_q : x^2 = A\}| - 1.$$

For a quadratic polynomial one obtains

$$[Ax^2 + Bx + C]_q \equiv \left(\frac{B^2 - 4AC}{q}\right) + \left(\frac{A^2}{q}\right) \mod q.$$

There exist useful identities to reduce Legendre symbols if the argument factorizes (F. Knop, OS). Often many more variables can be eliminated.

Modularity

We focus on primes p and are interested in graphs whose c_2-invariant is congruent modulo p to the Fourier coefficients of a modular form.
Consider a normalized Hecke eigenform f of weight w for a congruence subgroup of $SL_2(\mathbb{Z})$ of level ℓ, possibly with a non-trivial Dirichlet-character, with an integral Fourier expansion

$$f(\tau) = \sum_{k=0}^{\infty} a_k q^k, \quad q = \exp(2\pi \mathrm{i} \tau), \quad a_k \in \mathbb{Z}.$$

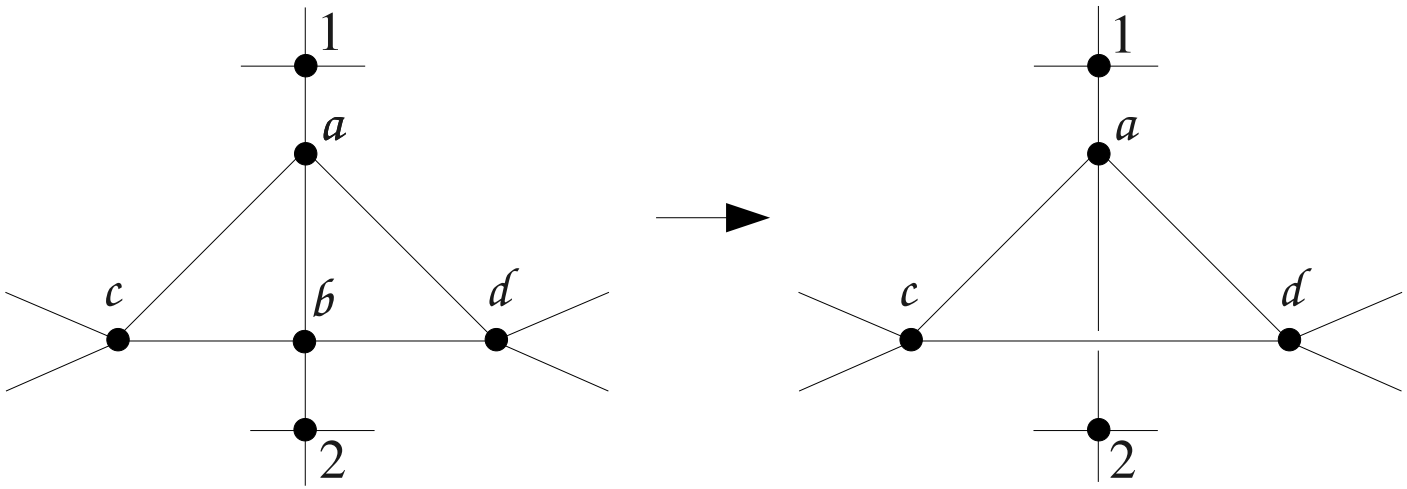
If the c_2-invariant of a graph G satisfies

$$c_2^{(p)}(G) \equiv -a_p \mod p$$

for all primes p, then c_2 is modular of weight w and level ℓ.
By modular correspondence the weight w is linked to the dimension $w-1$ of a variety.

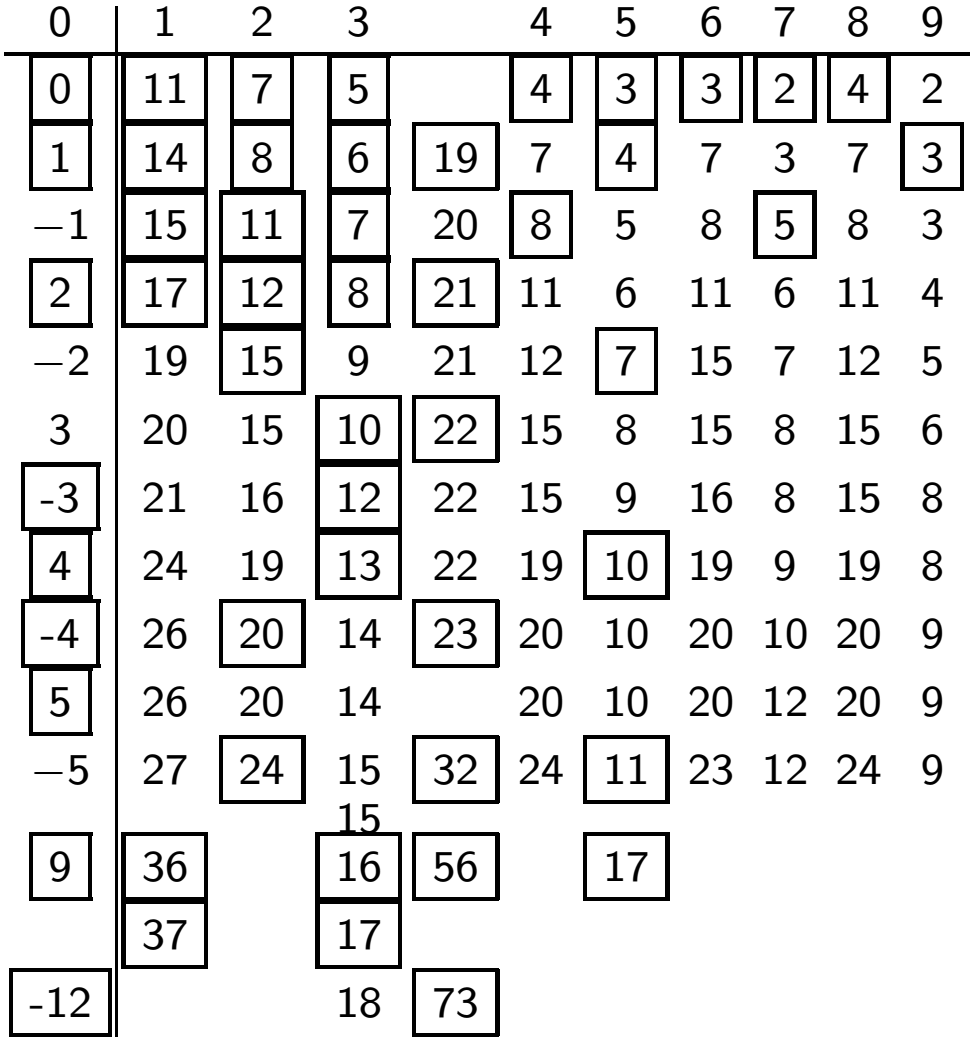
Graph reductions

In general, many different graphs may have equal c_2. In particular, the c_2 is invariant under completion and double triangle reductions:



With these reductions it was possible to study all graphs with $h_G \leq 10$ and all ϕ^4 graphs with $h_G \leq 11$. A partial analysis for ϕ^4 graphs was possible up to $h_G \leq 13$.
For large h_G most c_2 sequences could not be identified.
The following Legendre symbols and modular forms were found:

All graphs



0	1	2	3		4	5	6	7	8	9
0	11	7	5		4	3	3	2	4	2
1	14	8	6	19	7	4	7	3	7	3
−1	15	11	7	20	8	5	8	5	8	3
2	17	12	8	21	11	6	11	6	11	4
−2	19	15	9	21	12	7	15	7	12	5
3	20	15	10	22	15	8	15	8	15	6
-3	21	16	12	22	15	9	16	8	15	8
4	24	19	13	22	19	10	19	9	19	8
-4	26	20	14	23	20	10	20	10	20	9
5	26	20	14		20	10	20	12	20	9
−5	27	24	15	32	24	11	23	12	24	9
			15							
9	36		16	56		17				
	37		17							
-12			18	73						

ϕ^4 graphs

0	1	2	3		4	5	6	7	8	9
0	11	7	5		4	3	3	2	4	2
1	14	8	6	19	7	4	7	3	7	3
−1	15	11	7	20	8	5	8	5	8	3
2	17	12	8	21	11	6	11	6	11	4
−2	19	15	9	21	12	7	15	7	12	5
3	20	15	10	22	15	8	15	8	15	6
-3	21	16	12	22	15	9	16	8	15	8
4	24	19	13	22	19	10	19	9	19	8
-4	26	20	14	23	20	10	20	10	20	9
5	26	20	14		20	10	20	12	20	9
−5	27	24	15	32	24	11	23	12	24	9
			15							
9	36		16	56		17				
	37		17							
-12			18	73						

Conjectures

We may formulate the following conjectures:

- ϕ^4 geometries are sparse.
- Every graph with c_2 invariant -1 reduces by completion and double triangle reduction to K_4.
- The only extensions of MZVs in ϕ^4 theory are 2^{nd}, 4^{th}, 6^{th} roots of unity.
- There are no curves (weight 2 modular forms) in ϕ^4 theory (proved for $h_G \leq 11$ and level ≤ 2000).
- Every Feynman geometry of dimension ≤ 2 is modular.
- Every ϕ^4 geometry of dimension ≤ 3 is modular.
- In every fixed dimension there exist only finitely many ϕ^4 geometries.

Questions

The following questions arise:

- Why are ϕ^4 geometries constrained (if true)?
- Is it possible to mathematically describe ϕ^4 geometries?
- What is the number content of ϕ^4 geometries?
- Is there a connection between ϕ^4 geometries and motivic Galois theory?