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Causality prescription of the eikonal

Jung-Wook Kim [MPI for Gravitational Physics (Albert Einstein Institute)]

Works in collaboration with : Joon-Hwi Kim, Sangmin Lee [2405.17056]

Joon-Hwi Kim, Sungsoo Kim, Sangmin Lee (WIP, [241*.*****])

Warmup: a curious observation

- Context: eikonal as the scattering generator [Kim, JWK, Lee; Gonzo, Shi]

$$\mathcal{O}_f = e^{\{\chi,\bullet\}}[\mathcal{O}] = \mathcal{O} + \{\chi,\mathcal{O}\} + \frac{1}{2!}\{\chi,\{\chi,\mathcal{O}\}\} + \frac{1}{3!}\{\chi,\{\chi,\{\chi,\mathcal{O}\}\}\} + \cdots$$

  - Motivated from the N-matrix [Damgaard, Plante, Vanhove]
    - KMOC formula [Kosower, Maybee, O'Connell]

$$\langle\psi|\mathcal{O}_{\text{out}}|\psi\rangle = \langle\psi|S^\dagger\mathcal{O}_{\text{in}}S|\psi\rangle = \langle\psi|e^{-\frac{i}{\hbar}N}\mathcal{O}_{\text{in}}e^{+\frac{i}{\hbar}N}|\psi\rangle$$

    - As an operator equation (a "symmetry transform")

$$e^{-\frac{i}{\hbar}N}\mathcal{O}_{\text{in}}e^{+\frac{i}{\hbar}N} = (e^{-\frac{i}{\hbar}N})_{\text{adj}}[\mathcal{O}_{\text{in}}] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \underbrace{[N,[N,[N,\cdots,[N,}_{n\text{ times}}\mathcal{O}_{\text{in}}]]\cdots]]$$

  - Iterated brackets = *Causality cuts* [Kim, JWK, Lee; Kim, JWK, Kim, Lee (WIP)]

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$$e^{-\frac{i}{\hbar}N}\mathcal{O}_{\rm in}e^{+\frac{i}{\hbar}N} = (e^{-\frac{i}{\hbar}N})_{\rm adj}[\mathcal{O}_{\rm in}] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \underbrace{[N,[N,[N,\cdots,[N,}_{n\ \text{times}}\mathcal{O}_{\rm in}]]\cdots]]$$

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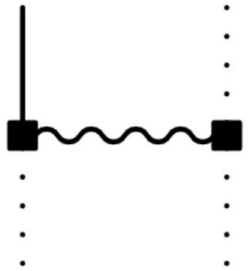
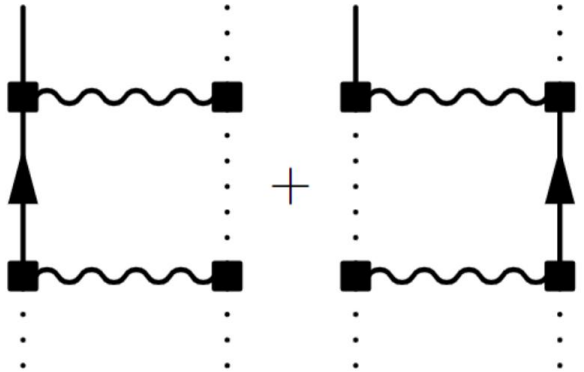
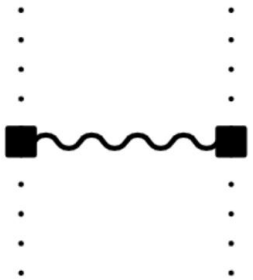
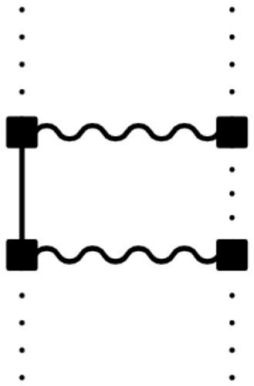
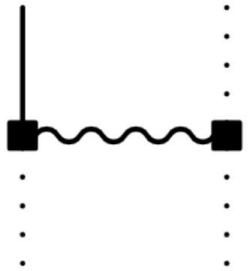
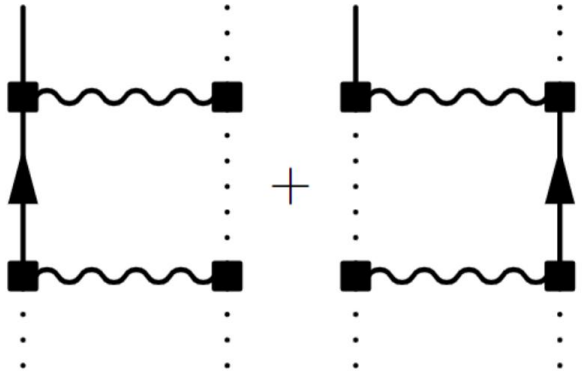
[Kim, JWK, Lee; Kim, JWK, Kim, Lee (WIP)]

$\Delta_{(1)} p_1^\mu =$  $= \{\chi_{(1)}, p_1^\mu\}$, $\Delta_{(2)} p_1^\mu =$ 

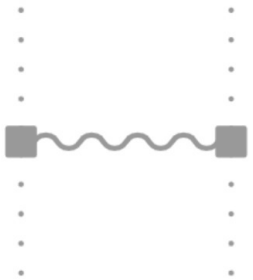
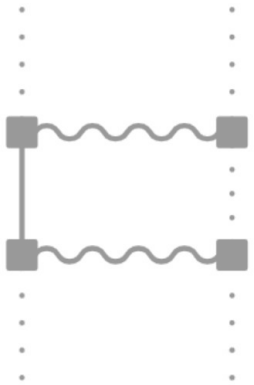
$i\chi_{(1)} =$  , $i\chi_{(2)} =$  $+ (1 \leftrightarrow 2)$

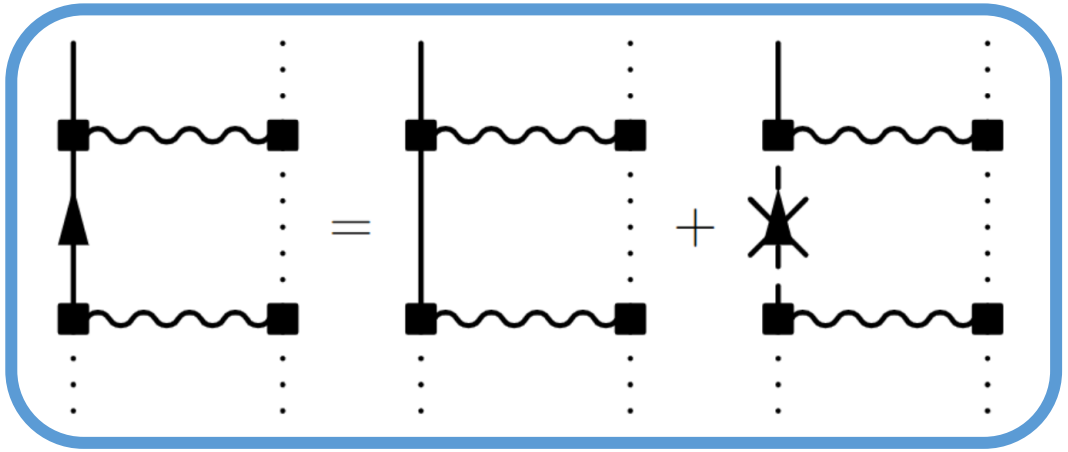
Warmup: a curious observation

[Kim, JWK, Lee; Kim, JWK, Kim, Lee (WIP)]

$\Delta_{(1)} p_1^\mu =$  $= \{\chi_{(1)}, p_1^\mu\}$, $\Delta_{(2)} p_1^\mu =$ 

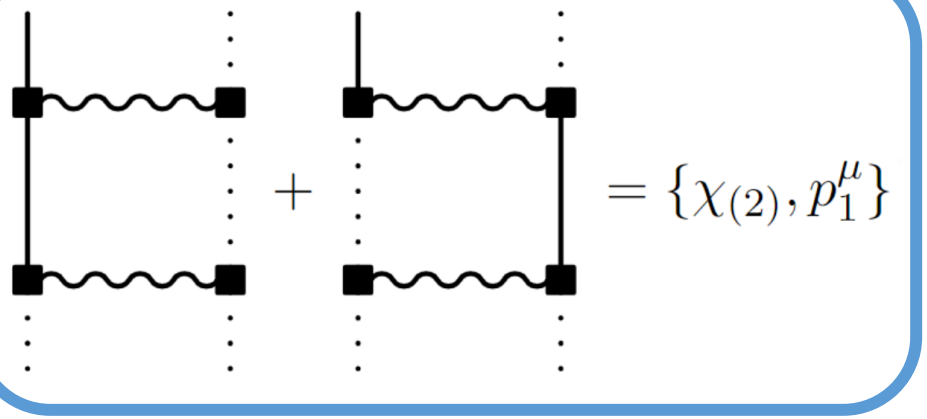
(ret.) = (sym.) + (causality cut)

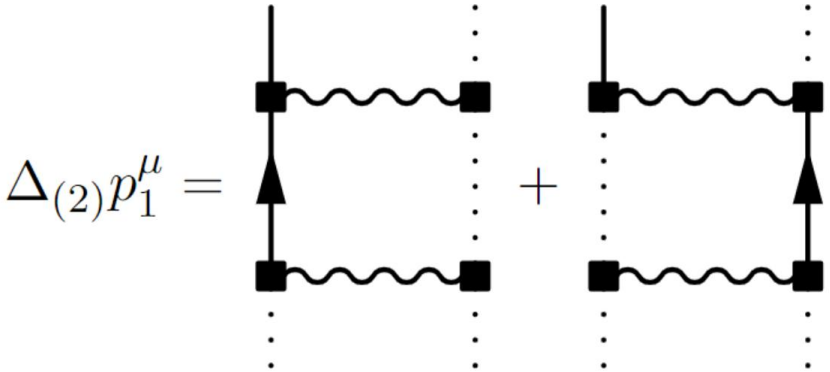
$i\chi_{(1)} =$  , $i\chi_{(2)} =$ 



Warmup: a curious observation

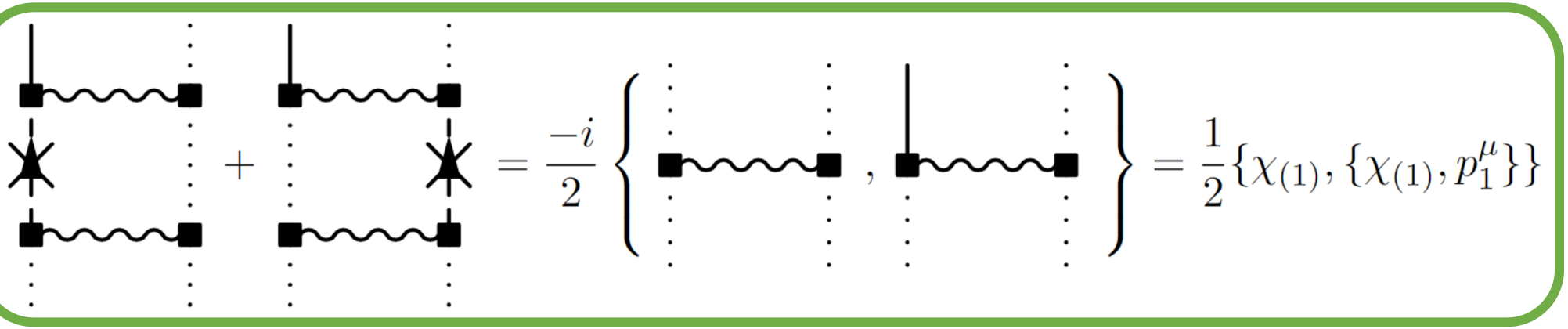
[Kim, JWK, Lee; Kim, JWK, Kim, Lee (WIP)]





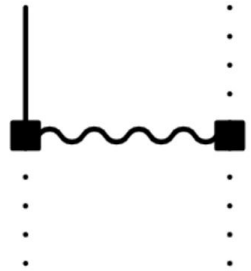
(sym.) = (eikonal)

(c. cut) = (nested P.B.)

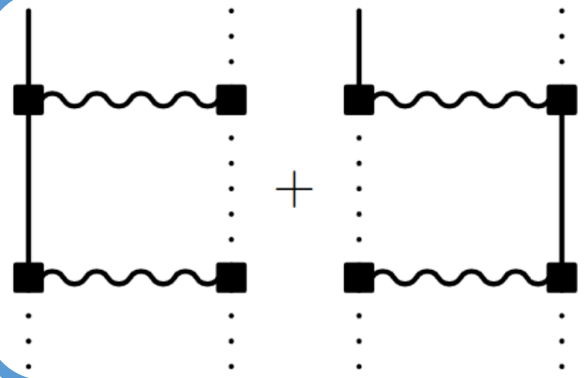


Warmup: a curious observation

[Kim, JWK, Lee; Kim, JWK, Kim, Lee (WIP)]

$\Delta_{(1)} p_1^\mu =$  $= \{\chi_{(1)}, p_1^\mu\}$

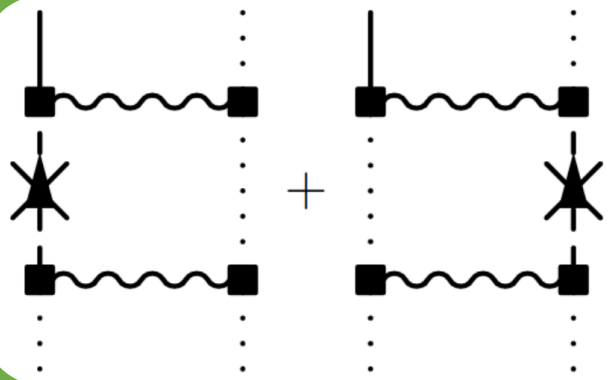
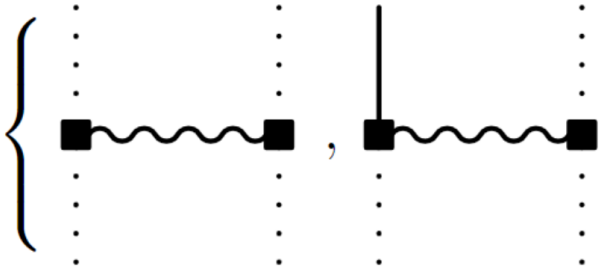
(1PL eikonal)

 $= \{\chi_{(2)}, p_1^\mu\}$

(2PL eikonal)

$$\mathcal{O}_f = \mathcal{O} + \{\chi, \mathcal{O}\} + \frac{1}{2!}\{\chi, \{\chi, \mathcal{O}\}\} + \cdots$$

(c. cut) = (iterated 1PL eikonal)

 $= \frac{-i}{2}$  $= \frac{1}{2}\{\chi_{(1)}, \{\chi_{(1)}, p_1^\mu\}\}$

Warmup: a curious observation

- How can $\{\chi_{(1)}, \Delta p_1^\mu\} \neq 0$? $\quad \chi_{(1)} \propto \log(b^2)\,,\ \Delta p_1^\mu \propto b^\mu / b^2$
- The impact parameter space is “noncommutative”
  - The definition of the impact parameter

$$\gamma = -(v_1 \cdot v_2)$$

$$b^\mu = \Delta x^\mu + (\Delta x \cdot v_1) w_1^\mu + (\Delta x \cdot v_2) w_2^\mu \qquad p_i^\mu = m_i v_i^\mu$$

$$\Delta x^\mu := x_1^\mu - x_2^\mu\,,\ w_1^\mu := \frac{\gamma v_2^\mu - v_1^\mu}{\gamma^2 - 1}\,,\ w_2^\mu := \frac{\gamma v_1^\mu - v_2^\mu}{\gamma^2 - 1} \qquad w_i \cdot v_j = -\delta_{ij}$$

  - The resulting Poisson bracket (up to mass-shell conditions)

$$\{b^\mu, b^\nu\} \doteq \frac{b^\mu w_1^\nu - b^\nu w_1^\mu}{m_1} - \frac{b^\mu w_2^\nu - b^\nu w_2^\mu}{m_2}$$

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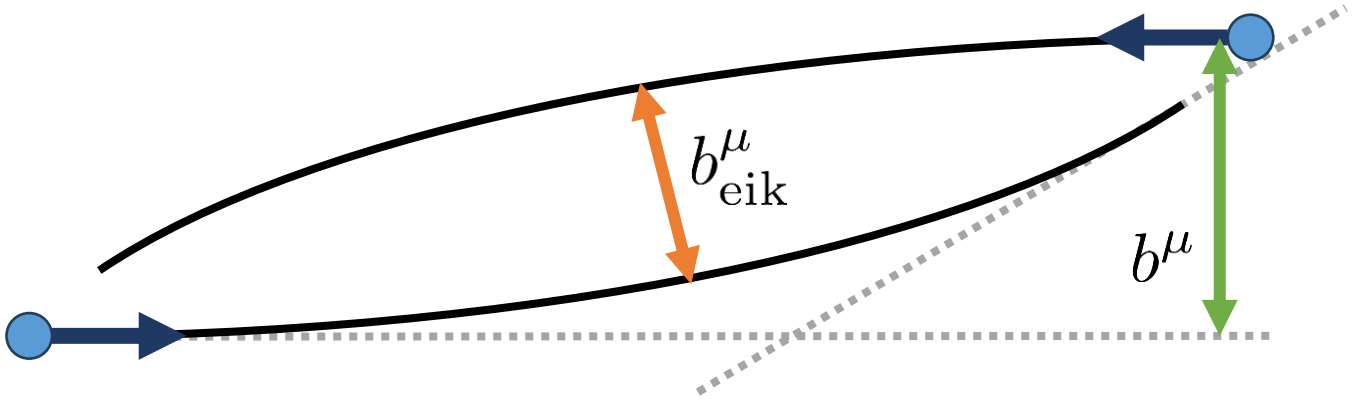
Warmup: a curious observation

- The iteration term computes the “longitudinal impulse”
  - Impulse along the incoming direction (required by energy conservation)

$$\{\chi_{(1)},\{\chi_{(1)},p_1^\mu\}\} = \left|\{\chi_{(1)},p_1^\mu\}\right|^2 \times \left(\frac{w_1^\mu}{m_1} - \frac{w_2^\mu}{m_2}\right) \quad \Rightarrow \quad p_{1\mu}\{\chi_{(1)},\{\chi_{(1)},p_1^\mu\}\} = -\left|\{\chi_{(1)},p_1^\mu\}\right|^2$$

  - Can be understood as “IPS rotation”
    - Relation to “KMOC cut” in one-loop waveform? [Georgoudis, Heissenberg, Russo]



Warmup: a curious observation

- The reorganisation extends to the spin kick
  - "Causality cut = Poisson bracket" in *any* Hamiltonian worldline model
  - Trivially applies to twistor worldlines (which includes spin DOF)
    - Caveat: regularisation needed for worldline 3pt vertices
  - Extends to SUSY WQFT (bosonic → Grassmannian)
- Deflection angle: no nested brackets! $\{J, J\} = 0 \;\Rightarrow\; \Delta\Phi = \{\chi, \Phi\} = -\dfrac{\partial\chi}{\partial J}$
- Another explanation for the $\mathcal{D}_{SL}$ operator [Bern, Luna, Roiban, Shen, Zeng; Luna, Moynihan, O'Connell, Ross]
  - Compensates neglection of $\{b^\mu, b^\nu\} \neq 0$

Talk overview

- Causality cuts & the eikonal in potential scattering
  - Causality cuts as Poisson brackets between subdiagrams
  - Reverse-engineer the eikonal from scattering generator equation

- From worldlines to gravitons through the Magnus expansion
  - What is the Magnus expansion?
  - The two-point functions in QFT
  - Example: 3PM 1SF eikonal in WQFT & 3PM impulse from the eikonal

- Summary and outlook

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Disclaimer

Lost, *never* Found: Factors of 2, π, i and minus signs.

image from [2103.17198]

The causality cut in potential scattering

- The worldline action
  - 1D target space; extension to general D straightforward

$$S = \int p dq - \frac{\kappa}{2}(p^2 + m^2) d\sigma + S_{\rm int}[p, q; A]$$

  - Background-fluctuation separation à la WQFT

$$p(\omega) = p_0 \delta(\omega) + \pi(\omega)\,,$$
$$q(\omega) = q_0 \delta(\omega) - i\kappa p_0 \delta'(\omega) + \phi(\omega)$$

  - Worldline fluctuation propagators (causality flow $[\omega' \to \omega] \Rightarrow [\omega + i0^+]$)

$$\langle \pi(\omega')\, \phi(\omega) \rangle = -\frac{1}{\omega} \delta(\omega' + \omega)\,, \quad \langle \phi(\omega')\, \phi(\omega) \rangle = \frac{i\kappa}{\omega^2} \delta(\omega' + \omega)$$

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mass shell condition

interaction potential (may depend on BG fields)

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The causality cut in potential scattering

- The causality cut
  - Compute the combination $[\omega' \to \omega] - [\omega \to \omega']$

$$\begin{aligned}
\langle \pi(\omega')\, \phi(\omega) \rangle \quad &\to \quad +i\delta(\omega')\delta(\omega) = -i\left(\delta(\omega')\left\{p_0, q_0\right\}\delta(\omega)\right) \,, \\
\langle \phi(\omega')\, \phi(\omega) \rangle \quad &\to \quad -\kappa\delta(\omega'+\omega)\delta'(\omega) = -i\left(\delta(\omega')\left\{q_0, p_0\right\}\left[-i\kappa\delta'(\omega)\right]\right. \\
&\qquad\qquad \left. + \left[-i\kappa\delta'(\omega')\right]\left\{p_0, q_0\right\}\delta(\omega)\right)
\end{aligned}$$

  - Vertex amputation rules: DC fluctuation ↔ BG derivative

$$\frac{\partial S_{\rm int}[p,q]}{\partial q_0} = \frac{\partial q(\omega)}{\partial q_0}\frac{\delta S_{\rm int}[p,q]}{\delta q(\omega)} = \delta(\omega)\frac{\delta S_{\rm int}[p,q]}{\delta \phi(\omega)}$$

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vertex rules w/ background derivatives

vertex rules attached to zero-freq (DC) fluctuations

The causality cut in potential scattering

- The causality cut as Poisson brackets
  - Sum over all causality cuts

$$\sum_{f,g=\pi,\phi} [\Delta_{f(\omega')} V_L] \langle f(\omega')\, g(\omega) \rangle [\Delta_{g(\omega)} V_R] \quad \to \sum_{f_0,g_0=p_0,q_0} \frac{\partial V_L}{\partial f_0} \times -i\{f_0, g_0\} \times \frac{\partial V_R}{\partial g_0} = -i\{V_L, V_R\}$$

  - Sum over all pairs of vertices from left/right subdiagrams

$$\sum_{V_a,V_b} \left(\frac{1}{S[G_{L\setminus a}]} \prod_{\substack{V_i \in G_L \\ V_i \neq V_a}} V_i \prod_{E_i \in G_L} E_i \right) \times -i\{V_a, V_b\} \times \left(\frac{1}{S[G_{R\setminus b}]} \prod_{\substack{V_i \in G_R \\ V_i \neq V_b}} V_i \prod_{E_i \in G_R} E_i \right) = -i\{I[G_L], I[G_R]\}$$

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augmentation operator (attach a fluctuation field to the vertex)

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edge Feynman rules

vertex Feynman rules

symmetry factor w/ vertex a coloured

subdiagram integrand

The eikonal from causality cuts

- Reverse-engineering the eikonal
  - Draw diagrams for the impulse (external leg → coloured vertex)
  - Subtract lower order eikonal iterations
  - Take the “antiderivative” of the diagrams (external leg ↔ vertex derivative)

- Examples:

$\chi_{(1)} = -$

$\Delta_{(1)}\vec{p} = -$

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$\chi_{(1)} =$

$\Delta_{(1)}\vec{p} =$

$$\{\chi_{(2)}, \vec{p}\} = \Delta_{(2)}\vec{p} - \frac{1}{2}\{\chi_{(1)}, \Delta_{(1)}\vec{p}\}$$

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  - Take the “antiderivative” of the diagrams (external leg ↔ vertex derivative)

- Examples:

$\chi_{(1)} =$

$\Delta_{(1)}\vec{p} =$

$$\{\chi_{(2)},\vec{p}\} = \Delta_{(2)}\vec{p} - \frac{1}{2}\{\chi_{(1)},\Delta_{(1)}\vec{p}\}$$

$\Delta_{(2)}p^{\mu} =$ 

$\{\chi_{(1)},\Delta_{(1)}\vec{p}\} = \{\bullet, \star\}$

$=$ 

The eikonal from causality cuts

- Reverse-engineering the eikonal
  - Draw diagrams for the impulse (external leg → coloured vertex)
  - Subtract lower order eikonal iterations
  - Take the “antiderivative” of the diagrams (external leg ↔ vertex derivative)

- Examples:

$\chi_{(1)} =$

$\Delta_{(1)}\vec{p} =$

$\{\chi_{(2)}, \vec{p}\} = \frac{1}{2}$  $= \left\{ \frac{1}{2}$ $, \vec{p} \right\}$

$\Delta_{(2)}p^{\mu} =$ 

$\{\chi_{(1)}, \Delta_{(1)}\vec{p}\} = \{$, $\}$

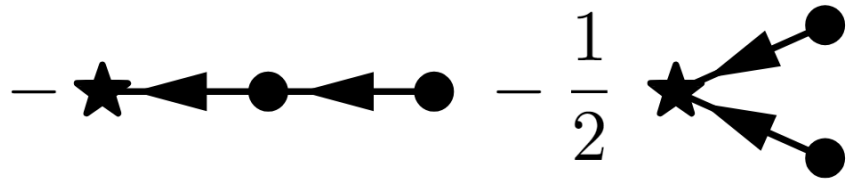
$=$ 

The eikonal from causality cuts

- More examples:

$$\{\chi_{(3)}, \vec{p}\}$$

$$= \underline{\Delta_{(3)}\vec{p}} - \frac{1}{2}\{\chi_{(1)}, \{\chi_{(2)}, \vec{p}\}\} - \frac{1}{2}\{\chi_{(2)}, \Delta_{(1)}\vec{p}\} - \frac{1}{6}\{\chi_{(1)}, \{\chi_{(1)}, \Delta_{(1)}\vec{p}\}\}$$

$\Delta_{(3)}\vec{p} =$  $- \frac{1}{2}$

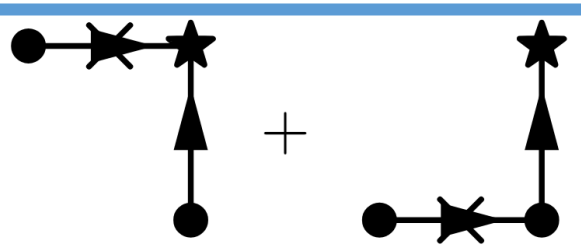
The eikonal from causality cuts

- More examples:

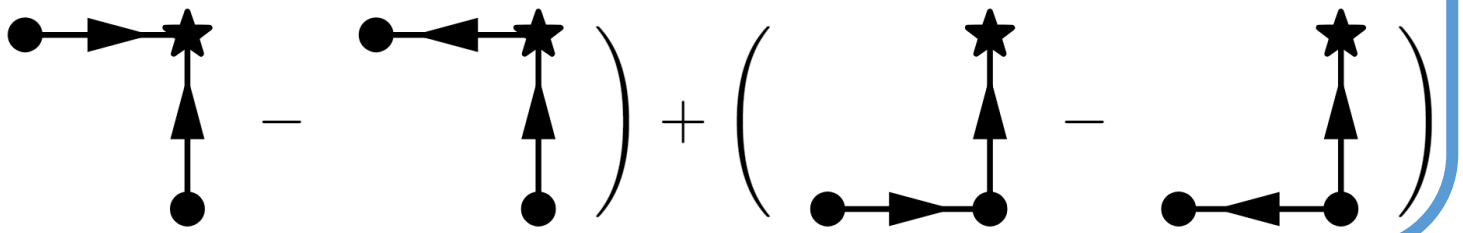
$$\{\chi_{(3)},\vec{p}\}$$

$$= \Delta_{(3)}\vec{p} - \frac{1}{2}\{\chi_{(1)},\{\chi_{(2)},\vec{p}\}\} - \frac{1}{2}\{\chi_{(2)},\Delta_{(1)}\vec{p}\} - \frac{1}{6}\{\chi_{(1)},\{\chi_{(1)},\Delta_{(1)}\vec{p}\}\}$$

$$-\{\chi_{(1)},\Delta_{(2)}\vec{p}\} = \Big\{ \bullet \ , \ \Big\} =$$



$$=$$



The eikonal from causality cuts

- More examples:

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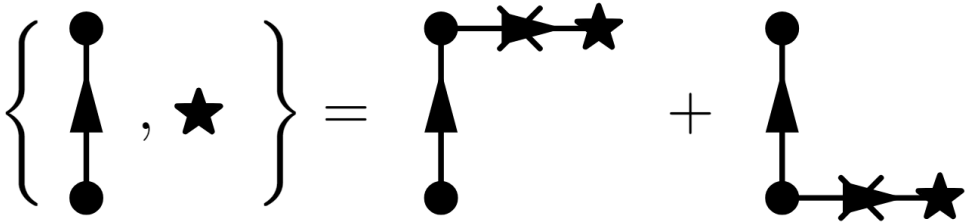
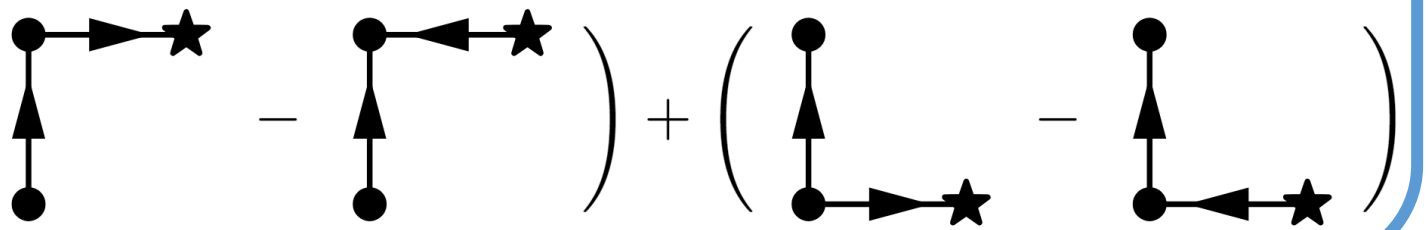
$-2\{\chi_{(2)},\Delta_{(1)}\vec{p}\} =$  $=$

$=$ 

The eikonal from causality cuts

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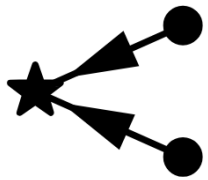
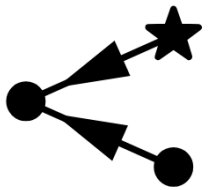
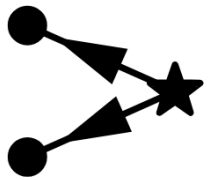
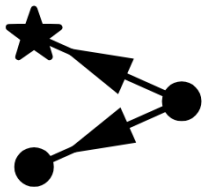
$$= \Delta_{(3)}\vec{p} - \frac{1}{2}\{\chi_{(1)},\{\chi_{(2)},\vec{p}\}\} - \frac{1}{2}\{\chi_{(2)},\Delta_{(1)}\vec{p}\} - \frac{1}{6}\{\chi_{(1)},\{\chi_{(1)},\Delta_{(1)}\vec{p}\}\}$$

$$= -\frac{1}{3}\Big($$  $$\Big)$$

$$- \frac{1}{12}\Big($$  $$+ 2$$  $$\Big) - \frac{1}{12}\Big($$  $$+ 2$$  $$\Big)$$

The eikonal from causality cuts

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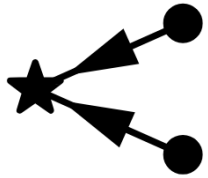
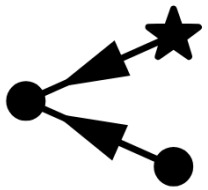
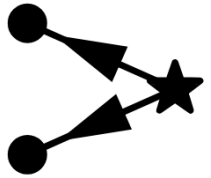
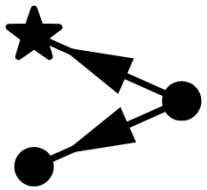
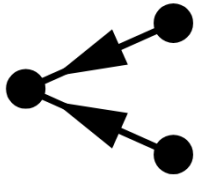
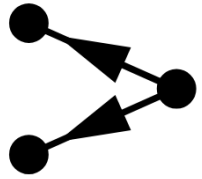
$$= -\frac{1}{3}\Big($$  $+$  $+$  $\Big)$

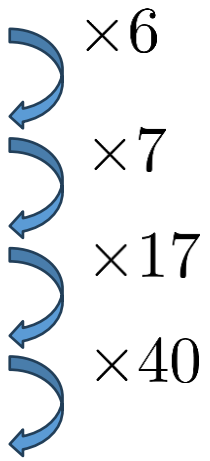
$$- \frac{1}{12}\Big($$  $+2$  $\Big) - \frac{1}{12}\Big($  $+2$  $\Big)$

$$= -\Big\{\frac{1}{3}$$  $+\frac{1}{12}\Big($  $+$  $\Big),\ \vec{p}\Big\}\ .$

The eikonal from causality cuts

- Computational complexity $\propto [n!]^2$

```
Chi[2] = chicompute[2]; // AbsoluteTiming        {0.042274, Null}
                          실제로 걸린 시간
Chi[3] = chicompute[3]; // AbsoluteTiming        {0.005847, Null}
                          실제로 걸린 시간
Chi[4] = chicompute[4]; // AbsoluteTiming        {0.029229, Null}
                          실제로 걸린 시간
Chi[5] = chicompute[5]; // AbsoluteTiming        {0.213203, Null}
                          실제로 걸린 시간
Chi[6] = chicompute[6]; // AbsoluteTiming        {3.35126, Null}
                          실제로 걸린 시간
Chi[7] = chicompute[7]; // AbsoluteTiming        {123.596, Null}
                          실제로 걸린 시간
Chi[8] = chicompute[8]; // AbsoluteTiming        ··· $RecursionLimit: Recursion depth of 1024 exceeded.
                          실제로 걸린 시간
                                                 {36.3835, TerminatedEvaluation[RecursionLimit]}
```



Machine: MacBook Pro 14 M1 Max
Mathematica version: 13.3

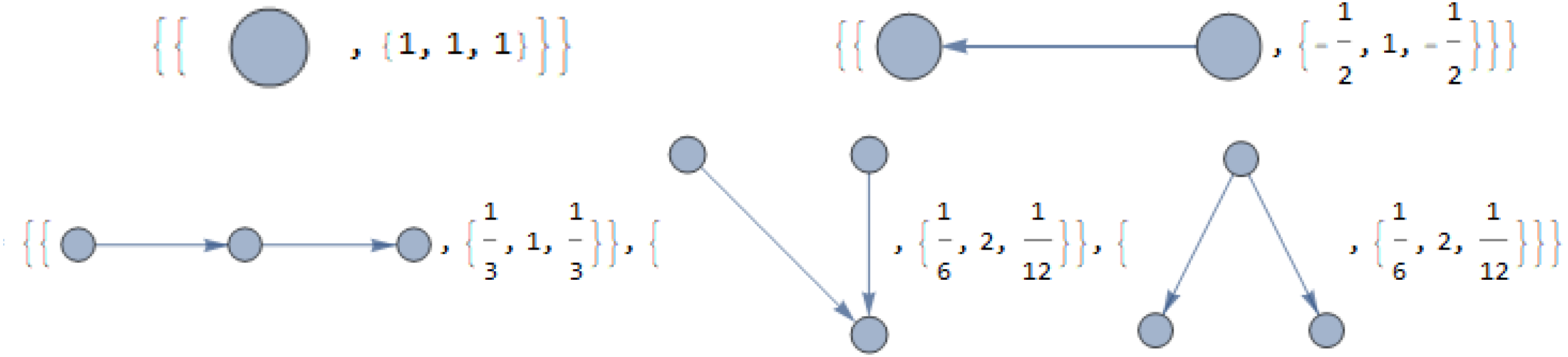
The eikonal from causality cuts

- Better to factor out symmetry factors (for multiple worldline species)
  - Order: $\left\{\omega(\tau), \sigma(\tau), \frac{\omega(\tau)}{\sigma(\tau)}\right\}$

our interest

symmetry factor

actual coefficient of the eikonal



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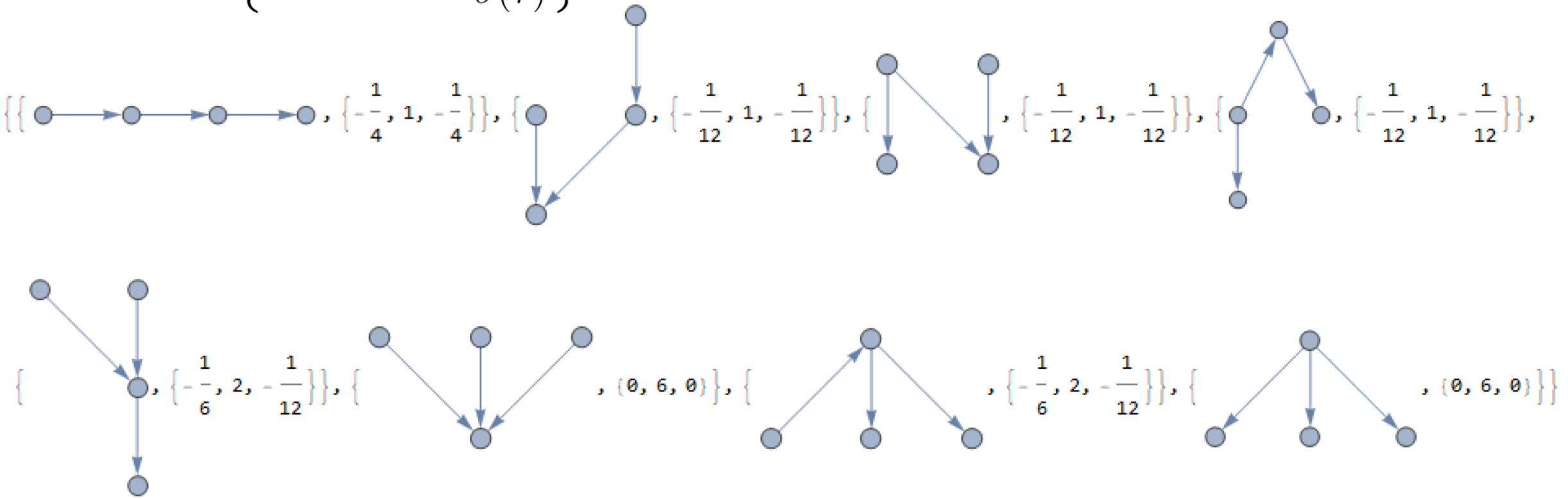

The Magnus expansion

- Can we improve causality-cut-based eikonal calculations?
  - Faster algorithm to determine $\omega(\tau)$ coefficients
  - Include graviton/photon propagators
  - “What are we actually computing?”

- This is answered by the **Magnus expansion**
  - Faster algorithm from Hopf algebra of graphs [Calaque, Ebrahimi-Fard, Manchon]
  - Generalisation to graviton/photon 2pt functions trivial
  - We *are* computing the N-matrix elements!

The Magnus expansion

- QFT 101: The S-matrix
  - Definition of the (transition) amplitude: $\mathcal{A}_{f\leftarrow i} = {}_{\text{out}}\langle f|i\rangle_{\text{in}} = {}_{\text{free}}\langle f|S|i\rangle_{\text{free}}$
  - S-matrix is the asymptotic limit of time evolution operator

$$S = \lim_{T\to\infty} U(+T;-T)\,, \quad |\psi(t_2)\rangle = U(t_2;t_1)|\psi(t_1)\rangle \;\Leftrightarrow\; \frac{dU(t_2;t_1)}{dt_2} = -iH_I(t_2)U(t_2;t_1)$$

  - Operational definition: the Dyson series of interaction Hamiltonian

$$\langle\psi_2|U(T;0)|\psi_1\rangle = \int_{\psi(0)=\psi_1}^{\psi(T)=\psi_2}[\mathcal{D}\psi]e^{iS[\psi]} = \int[\mathcal{D}\psi]\left(e^{-i\int_0^T H_I(t)dt}\right)e^{iS_{\text{free}}[\psi]}$$

$$\therefore\; S = U(+\infty;-\infty) = \mathcal{T}e^{-i\int_{-\infty}^{+\infty}H_I(t)dt} = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_1\cdots t_n}\mathcal{T}H_I(t_1)\cdots H_I(t_n)$$

The Magnus expansion

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time ordering operator

The Magnus expansion

- The Magnus expansion [Magnus 1954]
  - Find $\Omega(t_2;t_1)$ as the (series) solution to the (matrix) differential equation

$$\frac{d}{dt}Y(t) = A(t)Y(t)\,, \quad Y(t_1) = Y_0 \qquad \Rightarrow \qquad Y(t_2) = e^{\Omega(t_2;t_1)}Y_0$$

  - In our context, computes the N-matrix as a (power) series in $H_I(t)$

$$\frac{d}{dt}|\psi(t)\rangle = -iH_I(t)|\psi(t)\rangle\,, \qquad |f\rangle = S|i\rangle = e^{iN}|i\rangle$$

  - Why you may never have heard about it: somewhat inconvenient
    - Nonuniform integration domain, Bernoulli numbers, commutators, etc.

$$\Omega_1 = \int_{t_1} A_1\,, \quad \Omega_2 = \frac{1}{2}\int_{t_1>t_2}[A_1,A_2]\,, \quad \Omega_3 = \frac{1}{6}\int_{t_1>t_2>t_3}[A_1,[A_2,A_3]]+[[A_1,A_2],A_3]$$

$$\Omega_4 = \frac{1}{12}\int_{t_1>t_2>t_3>t_4}[[[A_1,A_2],A_3],A_4]+[A_1,[[A_2,A_3],A_4]+[A_1,[A_2,[A_3,A_4]]]+[A_2,[A_3,[A_4,A_1]]]$$

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2pt functions in perturbative QFT

- Causality cuts wrote the eikonal in retarded/advanced propagators
  - How do we get retarded/advanced propagators in the Magnus expansion?

- Review of 2pt function in QFT
  - Abstract field $\phi(x)$: stands for *any* field DOF of theory including worldlines
    - Assume Lagrangian (2nd order) action / dynamical (not Lagrange multiplier)
  - Basic building blocks: Wightman functions

$$G^+(x,y) := \langle 0|\phi(x)\phi(y)|0\rangle\,, \quad G^-(x,y) := \langle 0|\phi(y)\phi(x)|0\rangle$$

  - Wightman functions satisfy homogeneous wave equations

$$(\Box_x - m^2)\,G^\pm(x,y) = (\Box_y - m^2)\,G^\pm(x,y) = 0$$

for worldlines, $-\partial_\sigma^2$

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2pt functions in perturbative QFT

- Review of 2pt function in QFT
  - Pauli-Jordan function: commutator

$$G(x,y) := \langle 0|[\phi(x),\phi(y)]|0\rangle = G^{+}(x,y) - G^{-}(x,y) = [\phi(x),\phi(y)]$$

  - Retarded/advanced propagators: attach step function to P-J ftn

$$G_R(x,y) := +\theta(x^0 - y^0)\, G(x,y)$$

$$G_A(x,y) := -\theta(y^0 - x^0)\, G(x,y) = G_R(y,x)$$

  - Satisfies sourced wave equations
    - Use equal time commutation relation: $\pi(x) = \partial_0 \phi(x)$

$$(\Box_x - m^2)\, G_{R,A}(x,y) = -\delta(x^0 - y^0)\, [\partial_0\phi(x),\phi(y)] = i\delta^D(x-y)$$

2pt functions in perturbative QFT

- Review of 2pt function in QFT
  - Feynman propagator ≠ average of retarded & advanced
    - Equality for worldline propagators

$$G_F(x,y) := \langle 0|T\phi(x)\phi(y)|0\rangle = \theta(x^0-y^0)\,G^+(x,y) + \theta(y^0-x^0)\,G^-(x,y)$$
$$= \frac{1}{2}\left[G_R(x,y)+G_A(x,y)\right] + \frac{1}{2}\left[G^+(x,y)+G^-(x,y)\right]$$

  - Causality cut = Pauli-Jordan function = commutator
    - The causality cut computes Poisson brackets even for bulk fields (gravitons)!

$$G_{R-A}(x,y) := G_R(x,y) - G_A(x,y) = G(x,y) = [\phi(x),\phi(y)]$$

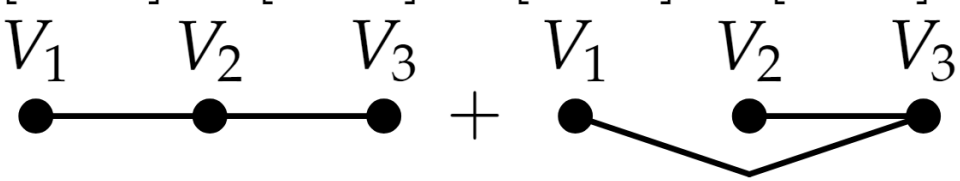
Eikonal from the Magnus expansion

- Diagrammar for Magnus expansion
  - Vertices ↔ interaction Hamiltonian/potential
  - Pauli-Jordan function ↔ undirected edges (orientation important)
  - Retarded/advanced Green's function ↔ directed edges (no orientation)

$$[V_1, V_2] = iG_{12} =$$

$$[V_1, [V_2, V_3]] = [iG_{12}] \times [iG_{23}] + [iG_{13}] \times [iG_{23}]$$

$$=$$



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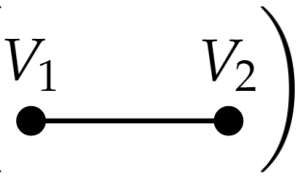
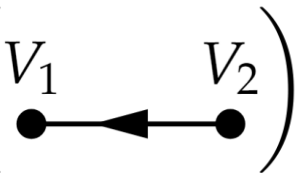
$$\chi_{(2)} = \frac{1}{2}\int_{t_1>t_2}[V_1, V_2] = \frac{1}{2}\int_{t_1,t_2}\theta(t_1-t_2)\times$$



$$= \frac{1}{2}\times$$



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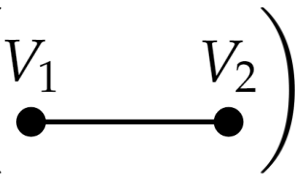
$$\chi_{(2)} = \frac{1}{2}\int_{t_1>t_2}[V_1, V_2] = \frac{1}{2}\int_{t_1,t_2}\theta(t_1-t_2)\times$$



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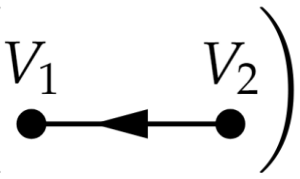
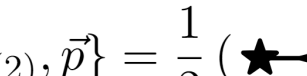


from slide 24: exact match!

$$\{\chi_{(2)}, \vec{p}\} = \frac{1}{2}\Big($$

$$+$$

$$\Big) = \Big\{\frac{1}{2}$$



$$, \vec{p}\Big\}$$

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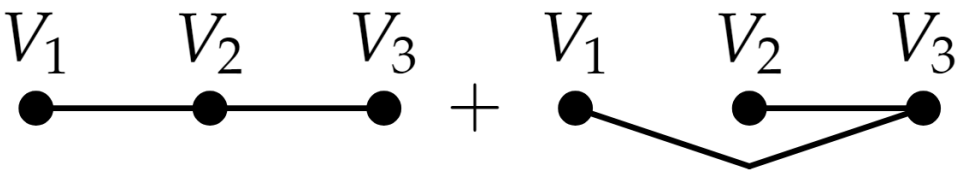

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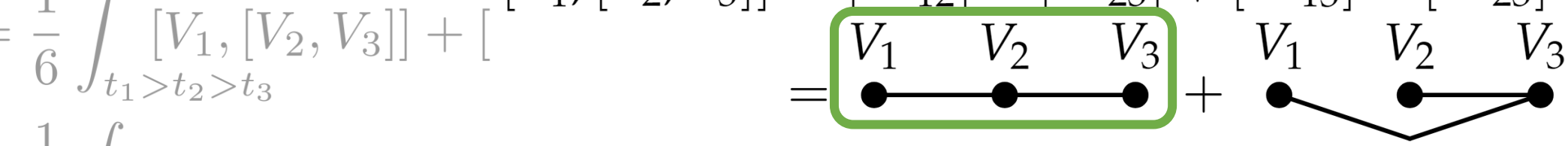

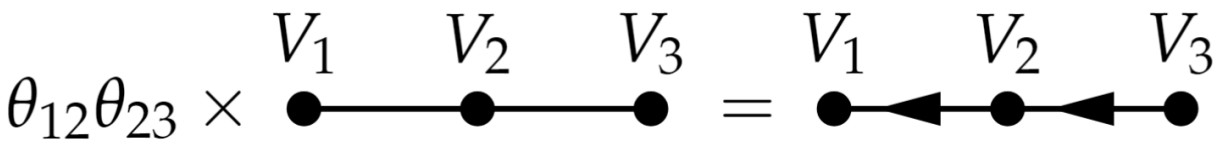

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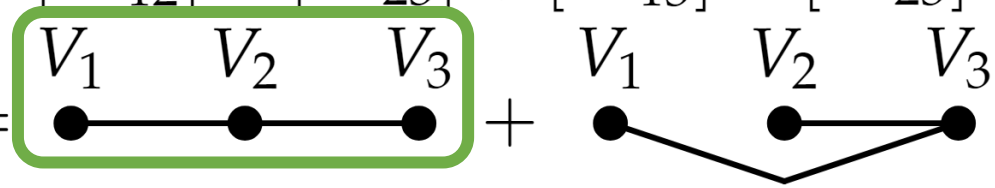
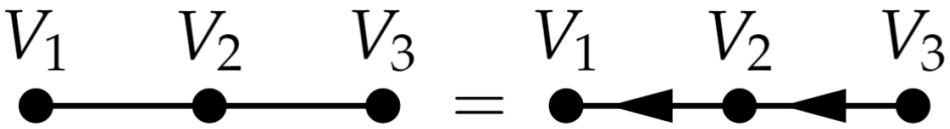
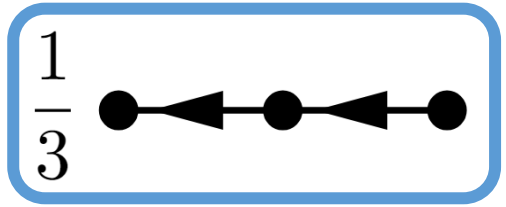


$$= \frac{1}{6}\int_{t_1,t_2,t_3} \theta(t_1-t_2)\theta(t_2-t_3)\times([V_1,[V_2,V_3]] + [[V_1,V_2],V_3])$$

$$\theta_{12}\theta_{23}\times \quad = \quad$$





from slide 29: exact match!

Eikonal from the Magnus expansion

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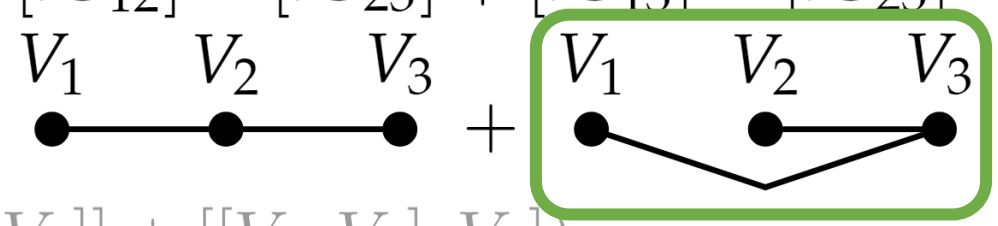

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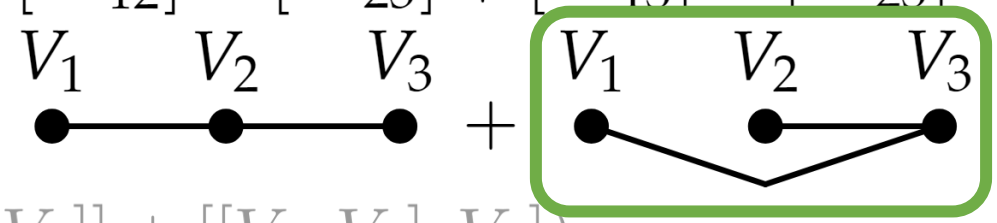
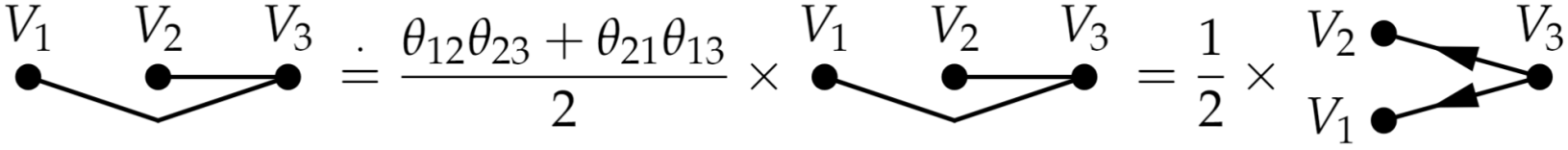


$$\theta_{12}\theta_{23}\times \quad \doteq \frac{\theta_{12}\theta_{23}+\theta_{21}\theta_{13}}{2}\times \quad = \frac{1}{2}\times$$



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$$\frac{1}{12}\left(\quad+\quad\right)$$

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Eikonal from the Magnus expansion

- Computed up to 8BI, consistency checked to 7BI
  - Direct computation is rather slow
  - Can we compute $\omega(\tau)$ at the level of graphs?

- Recursion from Hopf algebra of (tree) graphs [Calaque, Ebrahimi-Fard, Manchon]
  - Introduced for rooted trees, extension to non-rooted [Kim, JWK, Kim, Lee (WIP)]
  - Seems to recursively solve "operator differential equation" $e^{d\Omega} = \mathbb{1} + Adt$
  - Computational complexity $\propto (n!)$

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Magnus expansion coefficients

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Magnus expansion coefficients "exponentiation"

$$\omega(\tau) = -\sum_{p\in\mathcal{P}(\tau)\setminus\emptyset} e(p_\tau)\,\omega(\tau\backslash p)\,, \quad \omega(\bullet) = 1$$

sum over (proper) graph partitions

Useful properties of tree functions

- Edge contraction rule

$$\omega(\tau) + \omega(\tau|_{\bar{\mathfrak{e}}}) = -\omega(\tau_{\mathfrak{e}_c})$$

- Examples

Useful properties of tree functions

- Edge contraction rule

edge e reversed

$$\omega(\tau) + \omega(\tau|_{\bar{e}}) = -\omega(\tau_{e_c})$$

original diagram

edge e shrunk to a vertex

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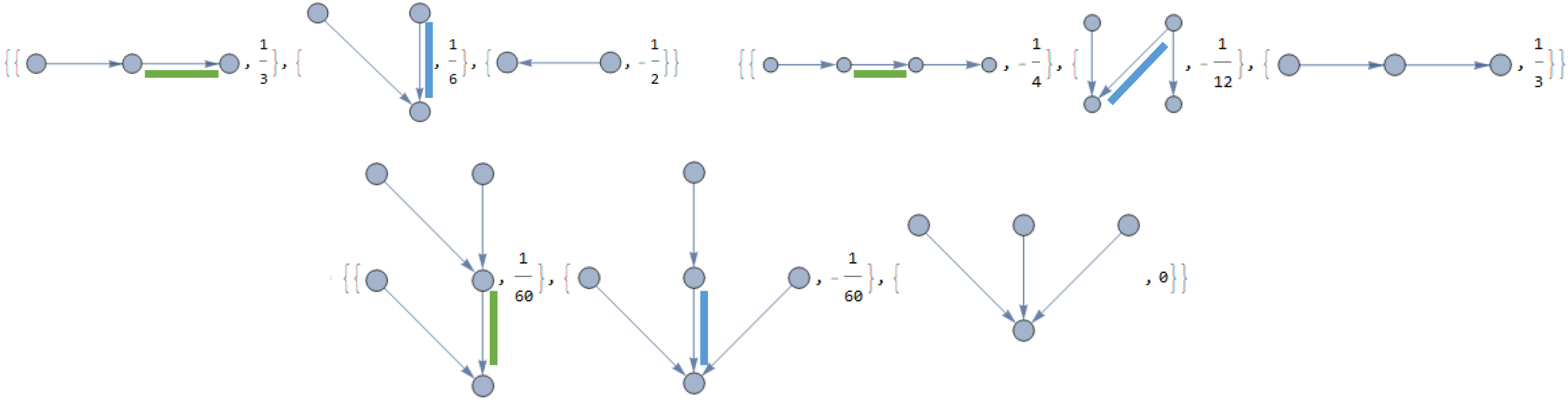

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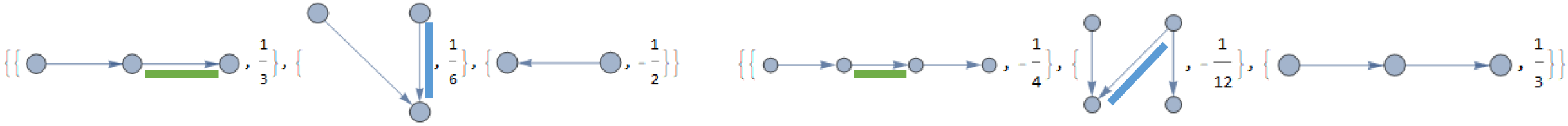

- Interpretation: “integrate out” heavy DOF of UV theory
  - Heavy DOF cannot go on-shell (causality prescription irrelevant)
  - Exchanging heavy DOF ↔ effective theory vertex

Useful properties of tree functions

- Feynman reduction

$$\sum_{\tau \in B(\hat{\tau})} \frac{\omega(\tau)}{\sigma(\tau)} = \frac{(-1)^{|\hat{\tau}|-1}}{S(\hat{\tau})}$$

- Examples

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Magnus expansion diagram coefficient

sum over all edge orientations

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Feynman diagram coefficient (inverse symmetry factor)

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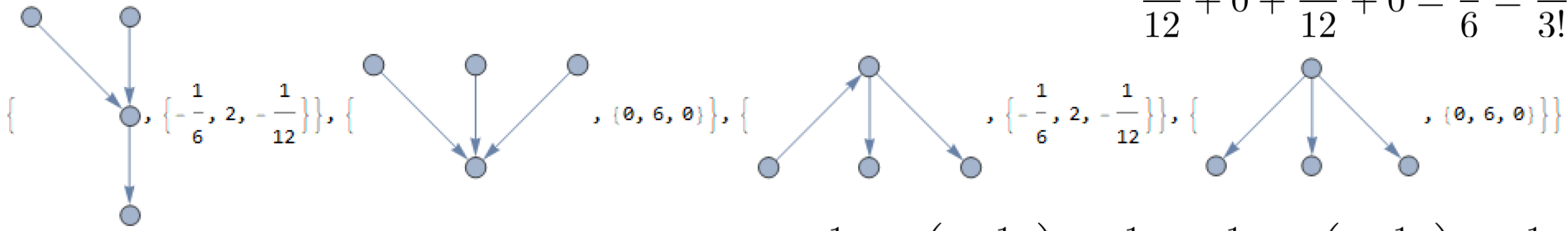
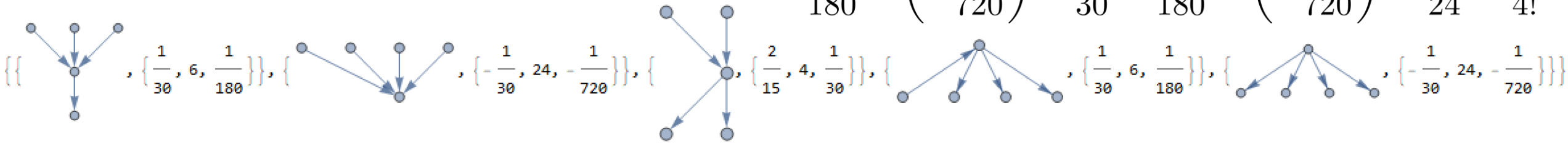
$$\frac{1}{12} + 0 + \frac{1}{12} + 0 = \frac{1}{6} = \frac{1}{3!}$$



$$\frac{1}{180} + \left(-\frac{1}{720}\right) + \frac{1}{30} + \frac{1}{180} + \left(-\frac{1}{720}\right) = \frac{1}{24} = \frac{1}{4!}$$



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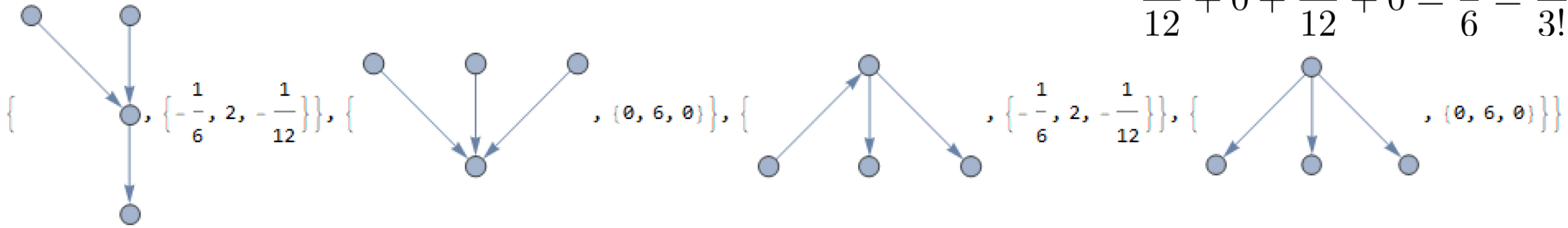

- Implies machineries for Feynman diagrams can be applied
  - E.g. generalised unitarity?

3PM 1SF WQFT eikonal

- Putting Magnus expansion under a stress test
  - 3PM: LO where Feynman/symmetric prescription breaks down
  - 1SF: LO where graviton causality prescription matters
  - Computed in amplitudes [Damgaard, Plante, Vanhove; Brandhuber, Chen, Travaglini, Wen]

- 3PM computations in WQFT [Jakobsen, Mogull; Jakobsen, Mogull, Plefka, Sauer]
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  - Why no eikonal?
    - Feynman/symmetric prescription IR divergent
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  - When computing the integrand, treat as Feynman diagram
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  - After IBP reduction, weight master integrals by Magnus expansion
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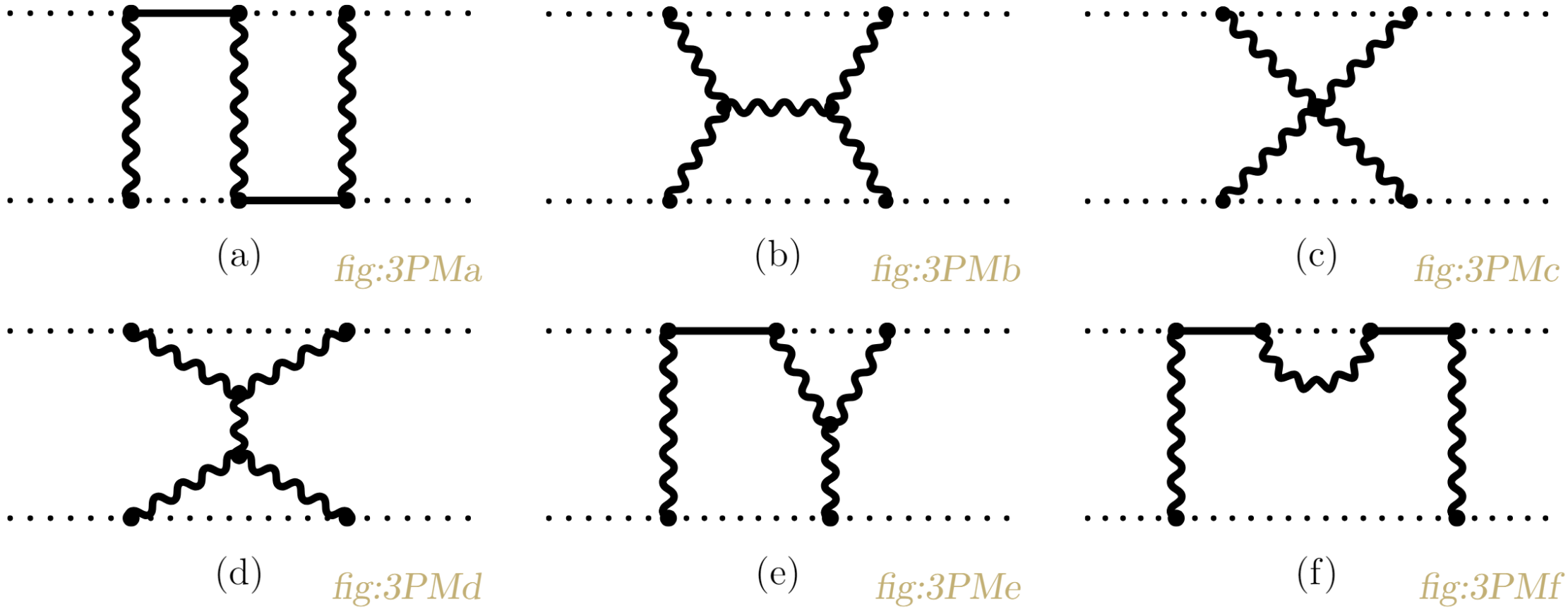
3PM 1SF WQFT eikonal

- Relevant diagrams



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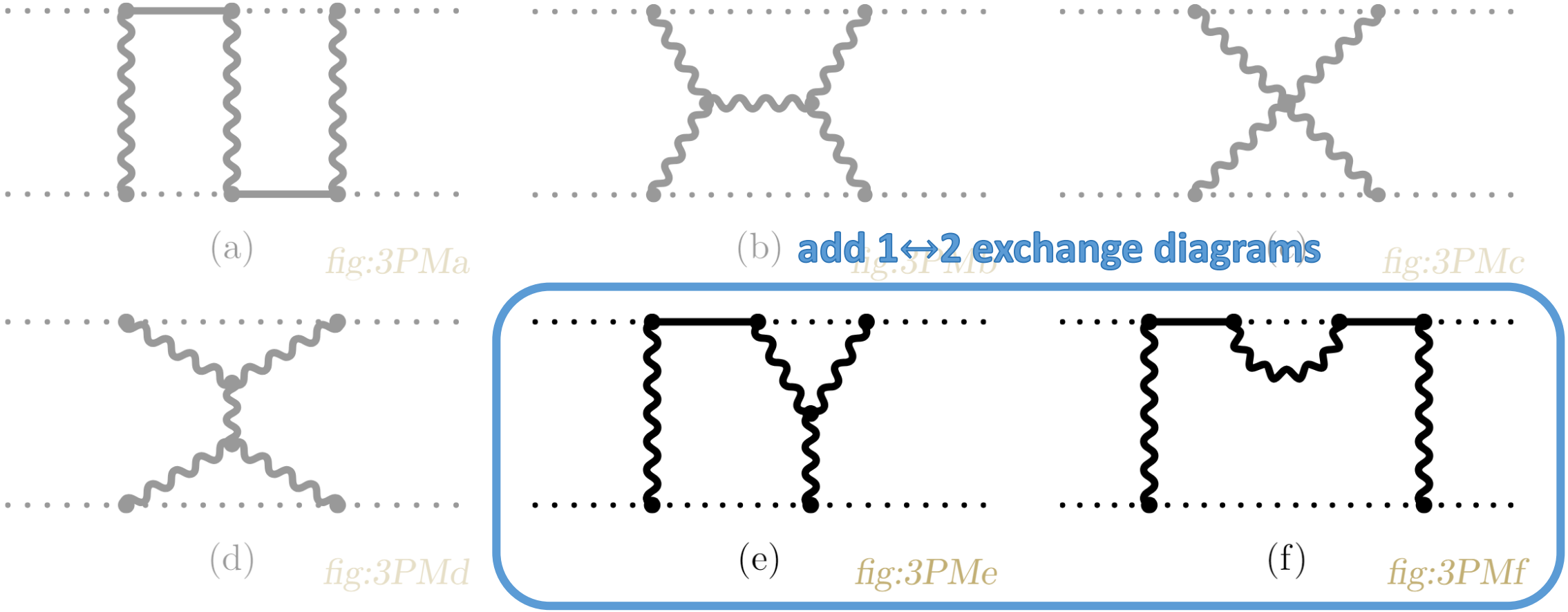
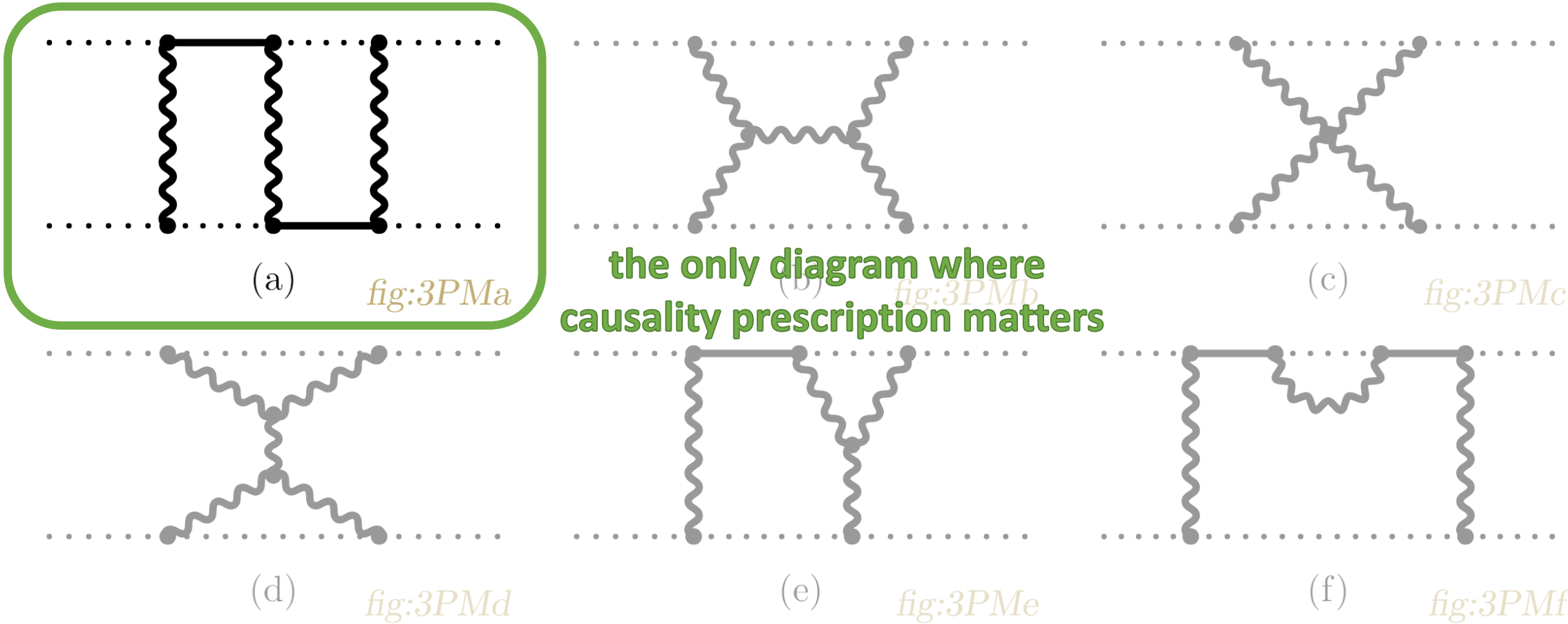


3PM 1SF WQFT eikonal

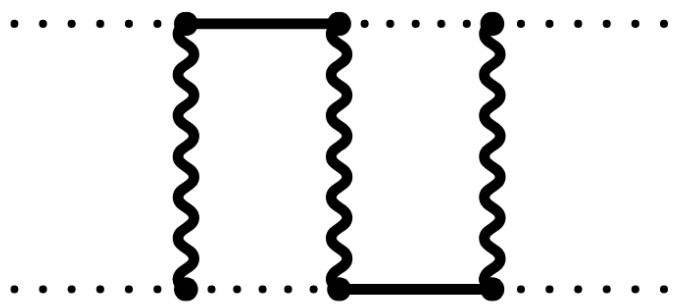
- Most interesting diagram



the only diagram where causality prescription matters

3PM 1SF WQFT eikonal

- Most interesting diagram: causality prescription matters!



(a)

fig:3PMa

$$I^{+\pm}_{(4)} = I^{+\pm}_{1,1,1,1,1,0,0} \propto \begin{cases} \dfrac{1}{2\epsilon^2} + \mathcal{O}(\epsilon^{-1}) & ++ \\ -\dfrac{1}{\epsilon^2} + \mathcal{O}(\epsilon^{-1}) & +- \end{cases}$$

$$I^{\sigma_1\sigma_2}_{n_1\cdots n_7} = \int_{l_1,l_2} \frac{\delta(l_1\cdot v_2)\delta(l_2\cdot v_1)}{\prod_{i=1}^{7} D_i^{n_i}} ,$$

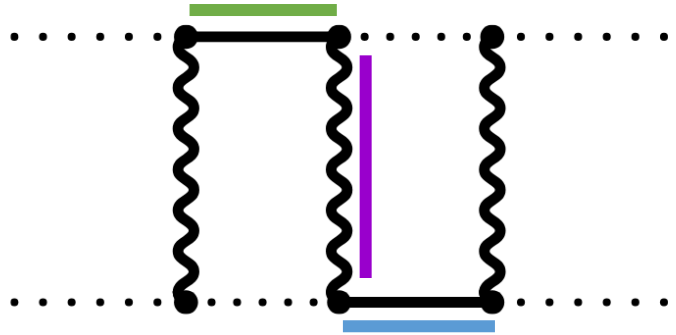
$$D_1 = l_1\cdot v_1 + \sigma_1 i0^+ , \qquad D_2 = l_2\cdot v_2 + \sigma_2 i0^+ , \qquad D_3 = (k^0 + i0^+)^2 - \vec{k}^2 ,$$

$$D_4 = l_1^2 , \qquad D_5 = l_2^2 ,$$

$$D_6 = (l_1+q)^2 , \qquad D_7 = (l_2+q)^2 .$$

3PM 1SF WQFT eikonal

- Most interesting diagram: causality prescription matters!



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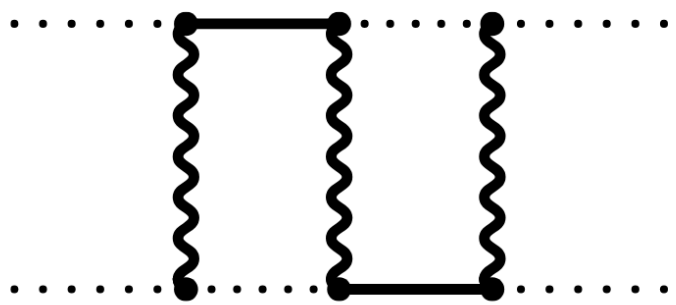

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- Symmetric causality prescription: $I_{(4)} = \frac{1}{2} I^{++}_{(4)} + \frac{1}{2} I^{+-}_{(4)} \propto -\frac{1}{4\epsilon^2} + \mathcal{O}(\epsilon^{-1})$
  - IR divergent(!) eikonal

$$\propto \frac{1}{\epsilon^2}(-q^2)^{2\epsilon} \simeq \frac{1}{\epsilon}\log(-q^2) \quad \Leftrightarrow \quad \propto \frac{1}{\epsilon}\frac{1}{|b|^2}$$

3PM 1SF WQFT eikonal

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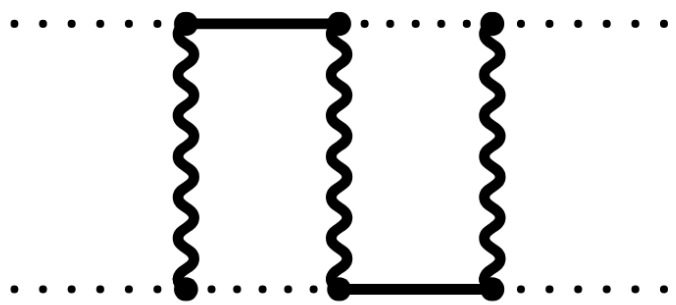

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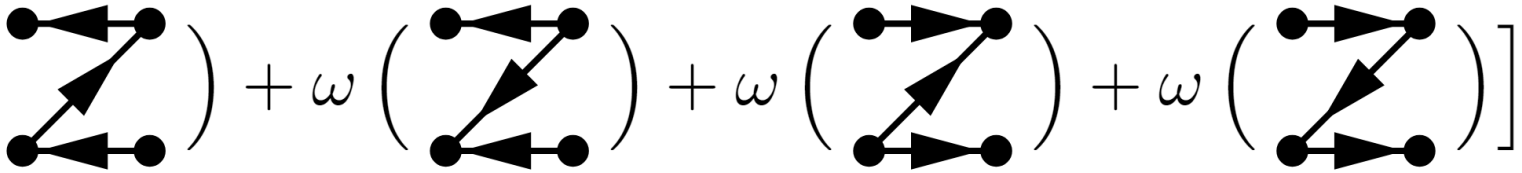
  - Exploit edge contraction: amputate outermost graviton propagators

3PM 1SF WQFT eikonal

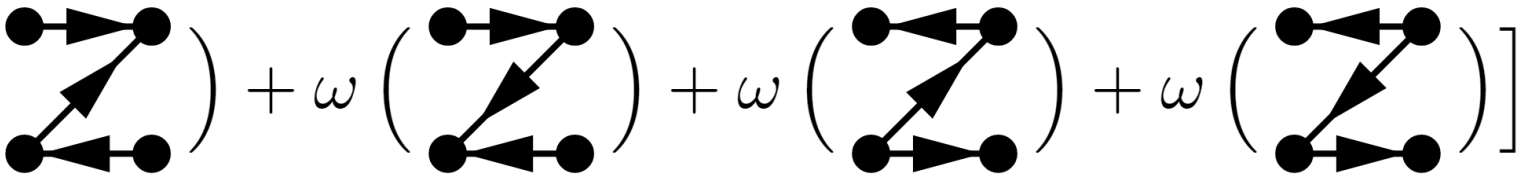
- Most interesting diagram: causality prescription matters!
  - ...
  - ...
- Mag
  - E

$$w_{++} = -\Big[\omega\Big(\;\Big) + \omega\Big(\;\Big) + \omega\Big(\;\Big) + \omega\Big(\;\Big)\Big]$$



$$= -\left[\left(-\frac{1}{4}\right) + \left(-\frac{1}{12}\right) + \left(-\frac{1}{12}\right) + \left(-\frac{1}{4}\right)\right] = \frac{2}{3}\,,$$

$$w_{+-} = -\Big[\omega\Big(\;\Big) + \omega\Big(\;\Big) + \omega\Big(\;\Big) + \omega\Big(\;\Big)\Big]$$



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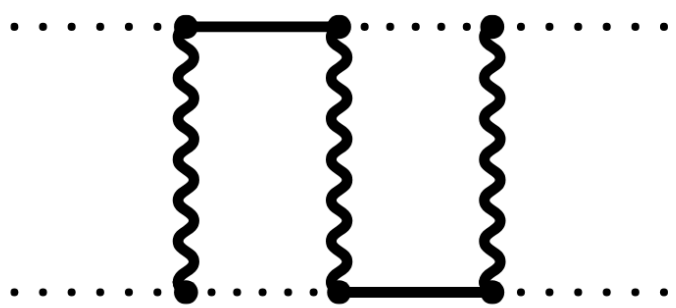

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  - IR finite eikonal! (shared feature of 3PM 0SF master integrals)

$$\propto \frac{1}{\epsilon}(-q^2)^{2\epsilon} \simeq \log(-q^2) \quad \Leftrightarrow \quad \propto \frac{1}{|b|^2}$$

3PM 1SF WQFT eikonal

- The computed eikonal is real
  - Matches (real part of) amplitude computations [Damgaard, Plante, Vanhove; Brandhuber, Chen, Travaglini, Wen]

$$\chi^{\rm 1SF}_{(3)} = \frac{G^3 m_1^2 m_2^2}{|b|^2}\left[\frac{2\gamma(-55+6\gamma^2(22-19\gamma^2+6\gamma^4))}{3(\gamma^2-1)^{5/2}} + \frac{4(3+12\gamma^2-4\gamma^4)\,\mathrm{arccosh}(\gamma)}{(\gamma^2-1)}\right.$$
$$\left. -\frac{2(1-2\gamma^2)^2(-8+5\gamma^2)}{3(\gamma^2-1)^2} + \frac{2\gamma(1-2\gamma^2)^2(-3+2\gamma^2)\,\mathrm{arccosh}(\gamma)}{(\gamma^2-1)^{5/2}}\right]$$

  - Only b-type master integrals appear (no v-type!)
    - Classification: which impulse direction (b^μ or v_i^μ) does it contribute to?
    - Consistent with $\{\chi_{(3)}, p_1^\mu\} \propto b^\mu$; "longitudinal impulse" from iterations
    - Reduced number of master integrals (no v-type master integrals; $12 \to 8$)

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rad. reac. terms

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3PM eikonal as scattering generator

- The (real) 3PM eikonal [Damgaard, Plante, Vanhove; Brandhuber, Chen, Travaglini, Wen]

$$\chi_{(1)} = -Gm_1m_2\frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}\log|b^2|\,,$$

$$\chi_{(2)} = +\frac{3\pi}{4}G^2m_1m_2(m_1+m_2)\frac{5\gamma^2-1}{\sqrt{\gamma^2-1}}\frac{1}{|b^2|^{1/2}}\,,$$

$$\chi_{(3)} = \frac{G^3m_1m_2}{3\sqrt{\gamma^2-1}|b^2|}\left[(m_1^2+m_2^2)X_{(3,0)}+m_1m_2(X_{(3,1c)}+X_{(3,1r)})\right]\,,$$

$$X_{(3,0)} = \frac{\left(64\gamma^6-120\gamma^4+60\gamma^2-5\right)}{\left(\gamma^2-1\right)^2}\,,$$

$$X_{(3,1c)} = \frac{2\gamma(36\gamma^6-114\gamma^4+132\gamma^2-55)}{(\gamma^2-1)^2}-\frac{12(4\gamma^4-12\gamma^2-3)}{\sqrt{\gamma^2-1}}\operatorname{arccosh}(\gamma)\,,$$

$$X_{(3,1r)} = (2\gamma^2-1)^2\left[-2\frac{(5\gamma^2-8)}{(\gamma^2-1)^{3/2}}+\frac{6\gamma(2\gamma^2-3)}{(\gamma^2-1)^2}\operatorname{arccosh}(\gamma)\right]\,.$$

3PM eikonal as scattering generator

- Comparison w/ 3PM impulse [Herrmann, Parra-Martinez, Ruf, Zeng; Jakobsen, Mogull, Plefka, Sauer; Kälin, Liu, Neef, Porto]
  - Misses (radiation loss) = (momentum lost by radiated graviton)

$$\Delta_{(3)}p_1^\mu = \Delta_{(3,c)}p_1^\mu + \Delta_{(3,rr)}p_1^\mu + \Delta_{(3,rl)}p_1^\mu$$

$$e^{\{\chi,\bullet\}}p_1^\mu\Big|_{\mathcal{O}(G^3)} = \Delta_{(3,c)}p_1^\mu + \Delta_{(3,rr)}p_1^\mu$$

$$\Delta_{(3,rl)}p_1^\mu = (v_2\cdot P_{\rm rad})w_2^\mu\,, \quad P^\mu_{\rm rad} := -\Delta_{(3)}p_1^\mu - \Delta_{(3)}p_2^\mu$$

[2207.00569]

```
(Humboldt3PM // ListToPlus) - impulse3PMsimp (* exactly radiaton loss term *)

γ (33 - 112 γ^2 + 165 γ^4 - 70 γ^6) G^3 m_1^2 m_2^2 π ArcCosh[γ] (u⊥)_1^i / (16 (-1 + γ^2)^2 (b_i b^i)^(3/2)) + (-5 + γ (76 + 5 γ (-30 + γ (12 + 7 γ)))) G^3 m_1^2 m_2^2 π Log[(1+γ)/2] (u⊥)_1^i / (8 Sqrt[-1 + γ^2] (b_i b^i)^(3/2)) +

(-1151 + γ (3336 + γ (-3148 + 3 γ (304 + γ (-113 + 2 (92 - 35 γ) γ))))) G^3 m_1^2 m_2^2 π (u⊥)_1^i / (48 ((-1 + γ^2) (b_i b^i))^(3/2))
```

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  - Radiation loss in "longitudinal" direction: $\chi_{(3)} \to \chi_{(3)} + \Delta\chi_{(3)}$ cannot work!
  - Radiation loss from "radiation eikonal" [Di Vecchia, Heissenberg, Russo, Veneziano] ?
    - E.g. as $\Delta_{(3,rl)}p_1^\mu \stackrel{?}{=} \frac{1}{2}\{\chi^{\rm rad}_{(1.5)},\{\chi^{\rm rad}_{(1.5)},p_1^\mu\}_\circ\}_\circ$ including graviton phase space?

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Summary

- The eikonal as a scattering generator: $\mathcal{O}_f = e^{\{\chi,\bullet\}}[\mathcal{O}]$
  - Generator of canonical transformation [in-state] → [out-state]
  - Causality cuts (ret. − adv.) compute Poisson brackets
  - Including graviton DOFs in the phase space seems necessary

- The Magnus expansion computes the eikonal
  - Causality prescription different from Feynman/symmetric (= Dyson series)
  - Reduced # of diagrams/master integrals compared to observables
  - IR finite result for the eikonal (Feynman prescription IR divergent)

Outlook

- Higher PM: graviton phase space? radiative eikonal?

- Application to amplitudes?
  - Many calculations are amplitude-based (we have more modern tools)
  - Can we compute the *effective Hamiltonian* from the Magnus expansion?
  - Relation to the BDS ansatz [Bern, Dixon, Smirnov] ?
  - IR divergences exponentiate [Weinberg]
    - Are N-matrix elements IR-finite (can we directly isolate IR divergences)?
    - If it does, does it extend to QCD? Can we use it for collider observables?
  - Need extension to loop diagrams (diagram → integrand hard!)