LFT Problem Set II:
RG Structure of the 4D Gauged NJL Model

Model and conventions. Consider an $SU(N_c)$ gauge theory with N_f Dirac fermions ψ (fundamental) and a four–fermion operator

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi} i \not{D}\, \psi + \frac{G}{\Lambda^2}\,\mathcal{O}_4, \qquad \mathcal{O}_4 \sim (\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi).$$

Define the *dimensionless* four–fermion coupling $\hat{G}(\mu) \equiv \mu^2 G_{\text{phys}}(\mu)$. Group factors: $C_A = N_c$, $C_F = \frac{N_c^2-1}{2N_c}$, $T_R = \frac{1}{2}$. Unless stated otherwise, assume a mass–independent scheme (e.g. $\overline{\text{MS}}$).

Problem 1. Canonical dimensions and allowed monomials

(a) Show that $\dim \mathcal{O}_4 = 6$ in $d = 4$ so $[G_{\text{phys}}] = -2$, and motivate $\hat{G} = \mu^2 G_{\text{phys}}$.

(b) By power counting and symmetries, list all polynomial monomials up to (overall) two-loop order that may appear in the beta functions

$$\beta_{\hat{G}} \equiv \mu \frac{d\hat{G}}{d\mu}, \qquad \beta_g \equiv \mu \frac{dg}{d\mu}.$$

Indicate their diagrammatic origin: canonical scaling, 4F bubbles, gauge dressing, gauge boxes, and two-loop composites.

Target structure (no coefficients):

$$\boxed{\beta_{\hat{G}} = 2\,\hat{G} + A\,\hat{G}^2 + B\,C_F\,g^2\hat{G} + C\,C_F^2\,g^4 + \underbrace{D\,\hat{G}^3}_{\text{2L}} + \underbrace{E\,g^2\hat{G}^2}_{\textbf{2L}} + \underbrace{F\,g^4\hat{G}}_{\text{2L}} + \underbrace{H\,g^6}_{\text{2L}} + \cdots}$$

$$\boxed{\beta_g = -b_0 g^3 - b_1 g^5 \; + \; \kappa_1\,\hat{G}\,g^3 \; + \; \kappa_2\,\hat{G}^2\,g^3 \; + \; \cdots}$$

with b_0, b_1 the universal pure–gauge coefficients. (See P2 about scheme dependence of κ_i.)

Problem 2. Gauge beta function and the lowest mixed terms

(a) Quote the universal pure–gauge form for vectorlike $SU(N_c)$ with N_f fundamentals:

$$\beta_g^{(\text{pure})} = -b_0 g^3 - b_1 g^5 + \mathcal{O}(g^7),\ b_0 = \frac{1}{(4\pi)^2}\left(\tfrac{11}{3}C_A - \tfrac{4}{3}T_R N_f\right),\ b_1 = \frac{1}{(4\pi)^4}\left(\tfrac{34}{3}C_A^2 - 4C_F T_R N_f - \tfrac{20}{3}C_A T_R N_f\right).$$

(b) Explain why in *mass–independent schemes* (e.g. $\overline{\text{MS}}$) insertions of the dimension–6 operator do *not* modify b_0, b_1 through two loops; equivalently $\kappa_{1,2} = 0$ to this order.

(c) Argue that in *cutoff/physical schemes* mixed terms

$$\beta_g \supset \kappa_1\, \hat{G}\, g^3 + \kappa_2\, \hat{G}^2\, g^3$$

can appear from gauge two–point renormalization with one or two 4F insertions; these are *scheme–dependent* and do not alter the universal pure–gauge coefficients.

Remark. If you write $\beta_{g^2} = \mu\, d(g^2)/d\mu$, the same content reads $\beta_{g^2} = -2b_0 g^4 - 2b_1 g^6 + \tilde{\kappa}_1 \hat{G} g^4 + \tilde{\kappa}_2 \hat{G}^2 g^4 + \cdots$.

Problem 3. Two–loop structures in $\beta_{\hat{G}}$ (including $g^2\hat{G}^2$)

Without computing coefficients, justify each of the following entries in $\beta_{\hat{G}}$:

- $2\hat{G}$ (engineering dimension).
- $A\,\hat{G}^2$ (one–loop 4F bubble).
- $B\,C_F\,g^2\hat{G}$ (gauge dressing of the 4F vertex / fermion legs).
- $C\,C_F^2\,g^4$ (gauge box / two–gluon exchange generating 4F).
- $E\,g^2\hat{G}^2$ (two–loop: gauge–dressed double 4F bubble, or a one–loop gauge correction inserted on a 4F bubble).
- $D\,\hat{G}^3$, $F\,g^4\hat{G}$, $H\,g^6$ (generic two–loop composites consistent with dimensions and symmetries).

Comment on expected *signs* (qualitatively) and on *scheme dependence* (most 4F coefficients are not universal).

Problem 4. Qualitative RG running and phase portraits

Consider

$$\beta_g = -b_0 g^3 - b_1 g^5 + \kappa_1 \hat{G} g^3 + \kappa_2 \hat{G}^2 g^3 + \cdots, \quad \beta_{\hat{G}} = 2\hat{G} + A\hat{G}^2 + BC_F g^2 \hat{G} + CC_F^2 g^4 + Eg^2 \hat{G}^2 + \cdots.$$

(a) **UV axis.** With $g \to 0$, discuss flows of $\hat{G}$ and conditions for an asymptotically safe NJL fixed point ($A < 0$). How do small g and the $g^2\hat{G}$ and $g^2\hat{G}^2$ terms perturb it?

(b) **Banks–Zaks vicinity.** If g has a weak IR fixed point g_*, analyze fixed points for $\hat{G}$ via $0 = 2\hat{G}_* + A\hat{G}_*^2 + BC_F g_*^2 \hat{G}_* + CC_F^2 g_*^4 + Eg_*^2 \hat{G}_*^2$. Sketch scenarios (one IR root vs none), and how E shifts the root.

(c) **Chiral symmetry breaking.** Explain qualitatively how larger g^2 (via γ_m) lowers the critical NJL channel; relate to trajectories crossing a separatrix in the $(g^2, \hat{G})$ plane.

The Mathematica command StreamPlot can be very helpful in identifying the RG flows of the system. You know how the couplings should behave around $g^2 = 0$ - that should help identifying the expected sign of the β functions.

Problem 5. HS (auxiliary scalar) viewpoint

Introduce an auxiliary scalar Φ with Yukawa y and quartic λ_Φ so that at tree level $\hat{G} \sim y^2/\lambda_\Phi$. Show how

$$\beta_y \supset y\,\gamma_\psi(g^2), \qquad \beta_{\lambda_\Phi} \supset -c_y y^4 + c_g g^4 + \cdots$$

map to the ($\hat{G}^2$, $g^2\hat{G}$, g^4) and ($g^2\hat{G}^2$) structures in $\beta_{\hat{G}}$.