Dr. Nicole Hartman
Dr. Jonas Würzinger
Nadine Bourriche

Tutorial Exercises Week 2

Problem 1

The Fisher Information is defined as $\mathbb{E}_x[(\partial_\theta \log p(x|\theta))^2]$

a) Derive an expression for the Fisher Information n identically random variables i.i.d from $p(x|\theta)$. What can you say about how information behaves as you add samples?

b) Derive a closed form expression for the Fisher Information $I(\lambda)$ for the Poisson distribution.

c) What is the Cramér-Rao bound on the variance of an estimator for the Poisson rate parameter.

Problem 2

We will derive a few relations for counting experiments:

a) Consider a case where instead of counting events, you are able to additionally measure for each event a discrete property with k possible values and associated probabilities p_a, where $a = 1, \ldots, k$ and $\sum_{a=1}^{k} p_a = 1$. Derive the probability of observing $n_1, \ldots, n_k$ events in each category given n events total.

b) Show that the joint probability of observing n events with category counts $n_1, \ldots, n_k$ and $\sum_i n_i = n$ can be expressed as a joint measurement of k individual Poisson processes. Derive their individual rates.

Problem 3

Implement the on/off problem $\mathrm{Pois}(n_1|\mu s + \gamma b)\,\mathrm{Pois}(n_2|\gamma\tau b)$ in code with $\tau = 5, s = 20, b = 50$

a) Given $n_1 = 105, n_2 = 265$, visualize the 2-D likelihood contour as a function of the parameters (s, b)

b) Perform a maximum likelihood fit to find the $\hat{\mu}, \hat{\gamma}$

c) Perform a maximum likelihood likelihood fit but for μ fixed at 1.5

d) Numerically compute the restricted MLE of the nuisance parameter as a function of the parameter of interest and add it to the 2-D likelihood plot .

e) Sample toy data from a model with $s = 1.5, b = 1.5$, perform maximum likelihood fits and observe the MLE distribution .

Problem 4

Consider the Gaussian model $p(x|\mu) = N(x|\mu, \sigma = 1)$

a) Produce Sampling Distributions for the likelihood ratio test for λ_{μ_0} for the null hypothesis $\mu' = \mu_0 = 0.5$ and the alternative hypothesis $\mu' = 1.5$ and visualize them.

b) What is the required threshold value in the test statistic in order to achieve a test of size 5%?

c) For an observation of $x = 1.5$, compute an approximate p-value based on the sampled data.

d) Decide whether the H_0 hypothesis should be rejected.

Problem 5

We continue with the setting in Problem 4

a) Derive the asymptotic variance of the MLE estimate of the parameter s.

b) Compute the non-centrality parameter for the alternative distribution.

c) Verify that the asymptotic sampling distributions describe the empirical data correctly.

d) What is the median expected p-value under the alternative ?

e) transform the test statistic data (both the sampling and observed data) to p-values and visualize them.

Problem 6

Take the 'on' model of $\text{Pois}(n|s + b)$ with known background $b = 3.0$. Assume you have observed 8 events

a) As a function of mu, what is the probability to observe data "less extreme"? At which point does this probability drop below 5%?

b) As a function of mu, what is the probability to observe data "more extreme"? At which point does this probability drop below 5%?

c) What is the equal tailed 95% confidence interval for the observed data?

d) What is the 95% upper limit on μ?

e) What is the maximum likelihood estimate of the parameter μ?