Tues morning, 23rd Sept 2025

Tutorial 2: Playing with probabilities

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1. Conjugate prior: Beta

Ex 16.2 from *Mathematical Statistics with Applications* by Wackerly, Mendenhall and Scheafer.

In lecture you found that prior $\text{beta}(\alpha, \beta)$ with a binomial likelihood, yields a beta posterior with new parameters $\alpha* = \alpha + k, \beta* = n - k + \beta$

Suppose you're an epidemiologist studying a rare disease with probability p.

You know the disease is rare (maybe $< p > \approx 0.25$), and you want to include the rarity of p in the statistical analysis that you're making.

Since you're psyched by Bayesian methods, you're gointg to consider p as a random variable with prior given by a Beta distribution.

1a) (Warm-up): What's the mean and variance of the beta distribtuion?

1b) For your epidemiology analyais, what α and β might you pick for the prior for $p(p)$?

1c) Plot this prior distribution that you're choosing

1d) Compare the prior and posterior distributions of the Bernoulli parameter p (proportion of responders to the new therapy) if we chose values for α and β given by the hypothetical data below:

a) $\alpha = 1, \beta = 3, n = 5, k = 2$
b) $\alpha = 1, \beta = 3, n = 25, k = 10$
c) $\alpha = 10, \beta = 30, n = 5, k = 2$
d) $\alpha = 10, \beta = 30, n = 25, k = 10$

2. Conguate priors: Gamma and Poisson

2a) Yesterday in the tutorial, we encountered the Poisson distribution, which is characterized by a length parameter, λ:

$$P(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Q for you: of the distributions that we've encountered so far… which would be the a good candidate for modelling the rate parameter, λ ?

2b) Consider n samples from the Poisson probability with outcomes $k_1, k_2, \ldots, k_n$.
For the prior you proposed in part (a), solve for the posterior probability.

Hint: if you define $k = \sum_{i=1}^{n} k_i$ to make the maths a bit more succinct

2c) Is the posterior in the same functional family as the prior??

2d) What is the Bayes MAP estimator for λ?
(**Helpful:** mode of the Gamma distribuion is $(\alpha - 1)\beta$)

2e) Show that this MAP estimator can be written as a weighted sum of (1) the mode from the prior and (2) the MLE of λ from the Poisson, $\hat{p}_{\mathrm{MLE}} = k/n$. Do the wieghts that you found make sense?